Supplement to the paper 'Empirical characteristic functions-based estimation and distance correlation for locally stationary processes'

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6 Proofs

Throughout this section C denotes a generic constant in $(0,\infty)$ that may change its value from line to line.

6.1 Proofs of Section 2

Proof of Lemma 2.1. (i) We abbreviate $v = \lceil |h/2| \rceil$ and define $\widetilde{\underline{X}}_t^{(v)}(u) = \underline{\mu}(u) + \sum_{|j|<v} A(u,j)\,\underline{\varepsilon}_{t-j}$. In the following we use the notation $g_{\underline{s}}(\underline{x}) := \exp(i\langle \underline{s}\,,\,\underline{x}\rangle)$. Now we obtain from independence of the innovations, as well as $|\mathrm{Cov}(X,Y)| \le E|XY| + E|X|\,E|Y|$ for complex-valued X, Y and $|g_{\underline{s}}(\underline{x})| = 1$,

$$\begin{aligned}
&\sup_{u_1,u_2\in[0,1]} \Big|\mathrm{Cov}\big(\exp\left(i\,\langle \underline{s}_1, \widetilde{\underline{X}}_h(u_1)\rangle\right), \exp\left(i\,\langle \underline{s}_2, \widetilde{\underline{X}}_0(u_2)\rangle\right)\big)\Big| \\
\le\ & \sup_{u_1,u_2\in[0,1]} \Big|\mathrm{Cov}\big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)) - g_{\underline{s}_1}(\widetilde{\underline{X}}_h^{(v)}(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_2))\big)\Big| \\
&+ \sup_{u_1,u_2\in[0,1]} \Big|\mathrm{Cov}\big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h^{(v)}(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_2)) - g_{\underline{s}_2}(\widetilde{\underline{X}}_0^{(v)}(u_2))\big)\Big| \\
\le\ & \sup_{u_1\in[0,1]} 2\,E\big|g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)) - g_{\underline{s}_1}(\widetilde{\underline{X}}_h^{(v)}(u_1))\big|_1 + \sup_{u_2\in[0,1]} 2\,E\big|g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_2)) - g_{\underline{s}_2}(\widetilde{\underline{X}}_0^{(v)}(u_2))\big|_1.
\end{aligned}$$

Now, we invoke a Taylor expansion of first order for $g_{\underline{s}}$. Note that the gradients with respect to $\underline{x}$ of both the real and the imaginary part of $g_{\underline{s}}(\underline{x})$ are bounded in $|\cdot|_1$-norm by $|\underline{s}|_1$. Hence, using (2.6), the first summand on the last right-hand side can be bounded by

$$\begin{aligned}
4\,|\underline{s}_1|_1 \sup_{u_1\in[0,1]} \big\|\widetilde{\underline{X}}_h(u_1) - \widetilde{\underline{X}}_h^{(v)}(u_1)\big\|_1 &\le 4\,|\underline{s}_1|_1 \sum_{|j|\ge v} \sup_{u_1\in[0,1]} |A(u_1,j)|_1 \cdot \big\|\underline{\varepsilon}_{h-j}\big\|_1 \\
&\le 4d\,|\underline{s}_1|_1\,\|\underline{\varepsilon}_0\|_1 \sum_{|j|\ge v} \frac{B}{l(j)}\,.
\end{aligned}$$

An analogous bound for the second summand on the right-hand side above finally yields

$$\sup_{u_1,u_2\in(0,1]} \Big|\mathrm{Cov}\big(\exp\left(i\,\langle \underline{s}_1, \widetilde{\underline{X}}_h(u_1)\rangle\right), \exp\left(i\,\langle \underline{s}_2, \widetilde{\underline{X}}_0(u_2)\rangle\right)\big)\Big| \le 4d\,(|\underline{s}_1|_1 + |\underline{s}_2|_1)\,\|\underline{\varepsilon}_0\|_1 \sum_{|j|\ge v} \frac{B}{l(j)}.$$

(ii) Using representations (2.1) and (2.5), Assumption 2.1, submultiplicativity of the $|\cdot|_1$-norm and the i.i.d. property of $(\underline{\varepsilon}_t)_t$ we get

$$\Big\|\underline{X}_{t,T} - \widetilde{\underline{X}}_t(t/T)\Big\|_1 \le \sum_{j=-\infty}^{\infty} |A_{t,T}(j) - A(t/T,j)|_1\ \big\|\underline{\varepsilon}_{t-j}\big\|_1 \le \frac{1}{T}\,\big\|\underline{\varepsilon}_0\big\|_1 \sum_{j=-\infty}^{\infty} \frac{B}{l(j)} = \mathcal{O}(T^{-1}).$$

(iii) All components of the mean function $\underline{\mu}$ are Lipschitz continuous due to the continuous differentiability condition in Assumption 2.1. Hence, we get

$$\big|\underline{\mu}(u_1) - \underline{\mu}(u_2)\big|_1 = \sum_{r=1}^{d} \big|\mu_r(u_1) - \mu_r(u_2)\big| \le Ld \cdot |u_1 - u_2|,$$

where L is the maximum of the d Lipschitz constants of the components of $\underline{\mu}(\cdot)$. Now, using representation (2.5) and a Taylor expansion for each entry $a^{(k,r)}(q,j)$ of matrix $A(q,j)$, it follows from Assumption 2.1(ii.2)

$$\begin{aligned}
\left\|\underline{\widetilde{X}}_h(u_1)-\underline{\widetilde{X}}_h(u_2)\right\|_1 &\leq \left|\underline{\mu}(u_1)-\underline{\mu}(u_2)\right|_1+\sum_{j=-\infty}^{\infty}\left|A(u_1,j)-A(u_2,j)\right|_1\left\|\underline{\varepsilon}_{h-j}\right\|_1\\
&= Ld\,|u_1-u_2|+\left\|\underline{\varepsilon}_0\right\|_1\cdot\sum_{j=-\infty}^{\infty}\max_{r=1,\ldots,d}\sum_{k=1}^{d}\left|\Big(\frac{\partial a^{(k,r)}(q,j)}{\partial q}\Big|_{q=\xi_{k,r,j}}\Big)(u_1-u_2)\right|\\
&\leq \Big(Ld+\left\|\underline{\varepsilon}_0\right\|_1\cdot\sum_{j=-\infty}^{\infty}\frac{Bd}{l(j)}\Big)\cdot|u_1-u_2|,
\end{aligned}$$

for suitable $\xi_{k,r,j}$ between $a^{(k,r)}(u_1,j)$ and $a^{(k,r)}(u_2,j)$. ■

Proof of Lemma 2.2. (i) By Lipschitz continuity of the function $\exp(i\cdot)$ with constant 1, we get from Lemma 2.1(ii)

$$\sup_{s\in[-S,S]^d,1\leq t\leq T}\left|\varphi_{t,T}(\underline{s})-\varphi\left(\frac{t}{T},\underline{s}\right)\right|\leq S\sup_{1\leq t\leq T}\left\|\underline{X}_{t,T}-\underline{\widetilde{X}}_t\left(\frac{t}{T}\right)\right\|_1\leq\left(S\,\|\underline{\varepsilon}_0\|_1\sum_{j=-\infty}^{\infty}\frac{B}{l(j)}\right)\frac{1}{T}\tag{6.1}$$

(ii) Let $k\geq 1$. Under Assumption 2.1(ii), we get existence of

$$\underline{\widetilde{X}}_1^{(l)}(u)=\frac{\partial^l}{\partial u^l}\left(\underline{\widetilde{X}}_1(u)\right)=\underline{\mu}^{(l)}(u)+\sum_{j=-\infty}^{\infty}A^{(l)}(u,j)\,\underline{\varepsilon}_{t-j},\quad l=0,\ldots,k.$$

Similarly, by Lebesgue's dominated convergence theorem and as $\exp(i\langle\underline{s},\underline{\widetilde{X}}_1(\cdot)\rangle)$ is a composition of k-times continuously differentiable functions, we get also existence and continuity of

$$\varphi^{(l)}(u,\underline{s})=\frac{\partial^l}{\partial u^l}\left(\varphi(u,\underline{s})\right)=E\left(\frac{\partial^l}{\partial u^l}\exp(i\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)\right),\quad l=0,\ldots,k.$$

To see this, we consider $l=1$ in detail first. Note that

$$\begin{aligned}
\frac{\partial e^{i\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle}}{\partial u} &= i\,e^{i\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle}\,\langle\underline{s},\underline{\widetilde{X}}_0^{(1)}(u)\rangle\\
&= \Big(-\sin(\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)+i\cdot\cos(\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)\Big)\,\langle\underline{s},\underline{\widetilde{X}}_0^{(1)}(u)\rangle,\quad u\in[0,1],
\end{aligned}$$

is absolutely integrable. Concerning the real part of the derivative, the mean value theorem gives

$$\begin{aligned}
&\lim_{h\to 0}\left|E\left(\frac{\cos(\langle\underline{s},\underline{\widetilde{X}}_0(u+h)\rangle)-\cos(\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)}{h}\right)-E\Big(-\sin(\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)\,\langle\underline{s},\underline{\widetilde{X}}_0^{(1)}(u)\rangle\Big)\right|\\
&=\lim_{h\to 0}\left|E\Big(-\sin(\langle\underline{s},\underline{\widetilde{X}}_0(u+\xi(h))\rangle)\,\langle\underline{s},\underline{\widetilde{X}}_0^{(1)}(u+\xi(h))\rangle\Big)-E\Big(-\sin(\langle\underline{s},\underline{\widetilde{X}}_0(u)\rangle)\,\langle\underline{s},\underline{\widetilde{X}}_0^{(1)}(u)\rangle\Big)\right|
\end{aligned}\tag{6.2}$$

for some $\xi(h)=\xi(h,\omega)\to 0$ as $h\to 0$ for each fixed ω. The right-hand side of (6.2) equals zero by Lebesgue's dominated convergence theorem. The imaginary part can be handled analogously.

Similarly, we also obtain continuity of $\varphi^{(l)}(\cdot, \underline{s})$.

For $l = 2$ we obtain absolute integrability of

$$\frac{\partial^2 e^{i\langle \underline{s}, \widetilde{\underline{X}}_0(u)\rangle}}{\partial u^2} = i\, e^{i\langle \underline{s}, \widetilde{\underline{X}}_0(u)\rangle} \langle \underline{s}, \widetilde{\underline{X}}_0^{(2)}(u)\rangle - \, e^{i\langle \underline{s}, \widetilde{\underline{X}}_0(u)\rangle} \left(\langle \underline{s}, \widetilde{\underline{X}}_0^{(1)}(u)\rangle\right)^2, \quad u \in [0,1],$$

from $E|\underline{\varepsilon}_0|_2^2 < \infty$. Hence, with the same arguments as before we can derive $\varphi^{(2)}(u, \underline{s})$. The proofs for higher order derivatives are analogous and therefore omitted. ■

6.2 Proofs of Section 3

Proof of Lemma 3.1. (i) *Bias – first part:*
For the bias term we get

$$\begin{aligned} E\big(\widehat{\varphi}(u, \underline{s}) - \varphi(u, \underline{s})\big) &= \frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T} - u\Big)\Big(\varphi_{t,T}(\underline{s}) - \varphi\Big(\frac{t}{T}, \underline{s}\Big)\Big) && (6.3)\\ &+ \frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T} - u\Big)\Big(\varphi\Big(\frac{t}{T}, \underline{s}\Big) - \varphi(u, \underline{s})\Big) && (6.4)\\ &+ \left(\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T} - u\Big) - 1\right)\cdot \varphi(u, \underline{s}). && (6.5) \end{aligned}$$

From Assumption 2.2, by standard arguments, we get that (6.5) is of order $\mathcal{O}(1/bT)$, uniformly over all $\underline{s}$ due to $|\varphi(u, \underline{s})| \leq 1$. Moreover, using Lemma 2.2(i), the term in (6.3) can be bounded in absolute value (uniformly in $\underline{s} \in [-S, S]^d$) by

$$\frac{C}{T}\sum_{t=1}^{T} \frac{1}{bT} K\Big(\frac{t/T - u}{b}\Big) = \frac{C}{T}\cdot \mathcal{O}(1) = \mathcal{O}(T^{-1}) \tag{6.6}$$

because of (2.9). Lemma 2.2 (ii) ensures existence of $\varphi^{(1)}(u, \underline{s})$, which is continuous in both arguments and therefore bounded on compact sets. These properties also hold true for both the real and the imaginary part of $\varphi^{(1)}(u, \underline{s})$. Hence, we can apply the mean value theorem to both parts and obtain for some constant $C < \infty$

$$\sup_{\underline{s}\in[-S,S]^d} \frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T} - u\Big)\Big|\varphi\Big(\frac{t}{T}, \underline{s}\Big) - \varphi(u, \underline{s})\Big| \leq C\cdot \sum_{t=1}^{T} \frac{1}{bT} K\Big(\frac{t/T - u}{b}\Big)\Big|\frac{t}{T} - u\Big| = \mathcal{O}(b)$$

due to (2.9). This implies (3.1) which completes the proof of part (i).

(ii) *Bias – second part:*
Following exactly the lines of the proof of (i) up to (6.6) yields

$$E\big(\widehat{\varphi}(u, \underline{s}) - \varphi(u, \underline{s})\big) = \frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T} - u\Big)\Big(\varphi\Big(\frac{t}{T}, \underline{s}\Big) - \varphi(u, \underline{s})\Big) + \mathcal{O}\Big(\frac{1}{bT}\Big). \tag{6.7}$$

We now split up (6.7) in its real and imaginary part and proceed separately. By Lemma 2.2(ii), we can use a Taylor expansion of the real part of $\varphi(\cdot, \underline{s})$ to derive for the corresponding part of the leading term in (6.7)

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\Big(\Re\varphi\Big(\frac{t}{T},\underline{s}\Big)-\Re\varphi(u,\underline{s})\Big)\\
=\;&\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\,\Re\varphi^{(1)}(u,\underline{s})\Big(\frac{t}{T}-u\Big)+\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\,\frac{1}{2}\Re\varphi^{(2)}(\widetilde{u},\underline{s})\Big(\frac{t}{T}-u\Big)^2 \qquad (6.8)
\end{aligned}$$

where $\widetilde{u}$ is between u and t/T. Now, as K is symmetric and Lipschitz, cf. Assumption 2.2, we get, for T sufficiently large, that

$$\begin{aligned}
&\Re\varphi^{(1)}(u,\underline{s})\cdot\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\Big(\frac{t}{T}-u\Big)\\
=\;&\Re\varphi^{(1)}(u,\underline{s})\cdot\sum_{t=1}^{T}\int_{(t-1)/T}^{t/T}\Big(K\Big(\frac{t/T-u}{b}\Big)\Big(\frac{t/T-u}{b}\Big)-K\Big(\frac{y-u}{b}\Big)\Big(\frac{y-u}{b}\Big)\Big)dy\\
=\;&\mathcal{O}(T^{-1})\,,
\end{aligned}$$

where the last equality holds since only $\mathcal{O}(bT)$ summands are nonzero. Using this bound for the first term in (6.8) shows that the real part of the bias (6.7) equals

$$\begin{aligned}
&\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\,\frac{1}{2}\Re\varphi^{(2)}(\widetilde{u},\underline{s})\Big(\frac{t}{T}-u\Big)^2+\mathcal{O}\Big(\frac{1}{bT}\Big)\\
=\;&\frac{1}{2}\Re\varphi^{(2)}(u,\underline{s})\cdot\int_0^1 K_b(y-u)\,(y-u)^2\,dy+\mathcal{O}\Big(\frac{1}{bT}\Big)\\
&+\frac{1}{2}\Re\varphi^{(2)}(u,\underline{s})\Big(\frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\Big(\frac{t}{T}-u\Big)^2-\int_0^1 K_b(y-u)\,(y-u)^2\,dy\Big) \qquad (6.9)\\
&+\frac{1}{2T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\Big(\frac{t}{T}-u\Big)^2\big(\Re\varphi^{(2)}(\widetilde{u},\underline{s})-\Re\varphi^{(2)}(u,\underline{s})\big). \qquad (6.10)
\end{aligned}$$

The expression in (6.9) can be bounded in absolute value by $\mathcal{O}(b/T)$. Lemma 2.2 (ii) ensures that $\Re\varphi^{(3)}(\cdot,\underline{s})$ is continuous and therefore bounded on $[0,1]$. Hence, using the mean value theorem, expression (6.10) can be bounded in absolute value by

$$\sup_{w\in[0,1]}|\Re\varphi^{(3)}(w,\underline{s})|\cdot\frac{1}{2T}\sum_{t=1}^{T} K_b\Big(\frac{t}{T}-u\Big)\Big(\frac{t}{T}-u\Big)^2\big|\widetilde{u}-u\big|=\mathcal{O}(b^3),$$

due to (2.9). Inserting the derived bounds into (6.9) and (6.10) yields that the real part of the bias term equals

$$\frac{1}{2}\Re\varphi^{(2)}(u,\underline{s})\int_0^1 K_b(y-u)\,(y-u)^2\,dy+\mathcal{O}\Big(\frac{1}{bT}\Big)+\mathcal{O}(b^3). \qquad (6.11)$$

By standard substitution, we get $\int_0^1 K_b(y-u)\,(y-u)^2\,dy = b^2\int_{-1}^1 K(z)\,z^2\,dz$. This, together with a completely analogous calculation for the imaginary part, gives assertion (ii).

(iii) *Covariances:*

For the covariances we show

$$bT\,\mathrm{Cov}(\widehat{\varphi}(u_1,\underline{s}_1),\widehat{\varphi}(u_2,\underline{s}_2)) = V((u_1,\underline{s}_1),(u_2,\underline{s}_2)) + o(1). \tag{6.12}$$

Using the notation $g_{\underline{s}}(\underline{x}) := \exp(i\langle \underline{s},\underline{x}\rangle)$ we immediately get

$$\begin{aligned} & bT\,\mathrm{Cov}(\widehat{\varphi}(u_1,\underline{s}_1),\widehat{\varphi}(u_2,\underline{s}_2)) \\ = \; & \frac{b}{T}\sum_{t_1,t_2=1}^{T} K_b\Big(\frac{t_1}{T}-u_1\Big)K_b\Big(\frac{t_2}{T}-u_2\Big)\mathrm{Cov}\big(g_{\underline{s}_1}(\underline{X}_{t_1,T}),g_{\underline{s}_2}(\underline{X}_{t_2,T})\big). \end{aligned} \tag{6.13}$$

In order to replace $\underline{X}_{t_i,T}$ by $\underline{\widetilde{X}}_{t_i}(\frac{t_i}{T})$ on the right-hand side, we first show

$$\mathrm{Cov}\big(g_{\underline{s}_1}(\underline{X}_{t_1,T}),g_{\underline{s}_2}(\underline{X}_{t_2,T})\big) = \mathrm{Cov}\big(g_{\underline{s}_1}(\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_{t_2}(\frac{t_2}{T}))\big) + \mathcal{O}(T^{-1}) \tag{6.14}$$

uniformly for all t_1,t_2. Using a Taylor expansion of first order for the real and imaginary part of $g_{\underline{s}_1}$ we get

$$\begin{aligned} \mathrm{Cov}\big(g_{\underline{s}_1}(\underline{X}_{t_1,T}),g_{\underline{s}_2}(\underline{X}_{t_2,T})\big) \;=\; & \mathrm{Cov}\big(g_{\underline{s}_1}(\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{X}_{t_2,T})\big) \\ & +\mathrm{Cov}\Big(\nabla\Re g_{\underline{s}_1}(\underline{\xi}_1)'\,(\underline{X}_{t_1,T}-\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{X}_{t_2,T})\Big), \\ & +i\cdot\mathrm{Cov}\Big(\nabla\Im g_{\underline{s}_1}(\underline{\xi}_2)'\,(\underline{X}_{t_1,T}-\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{X}_{t_2,T})\Big), \end{aligned} \tag{6.15}$$

where $\underline{\xi}_1$ and $\underline{\xi}_2$ are both between $\underline{X}_{t_1,T}$ and $\underline{\widetilde{X}}_{t_1}(t_1/T)$. Noting that $\nabla\Re g_{\underline{s}_1}(\underline{x}) = -\sin(\langle\underline{s}_1,\underline{x}\rangle)\cdot\underline{s}_1$ and $\nabla\Im g_{\underline{s}_1}(\underline{x}) = \cos(\langle\underline{s}_1,\underline{x}\rangle)\cdot\underline{s}_1$, as well as $|g_{\underline{s}_2}(\cdot)| = 1$, the second and the third summand on the right-hand side of (6.15) can each be bounded in absolute value by

$$2\,|\underline{s}_1|_1\cdot\Big\|\underline{X}_{t_1,T}-\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})\Big\|_1 = \mathcal{O}(T^{-1})$$

uniformly in t_1,t_2, due to Lemma 2.1 (ii). With exactly the same calculation the second argument can be replaced which yields (6.14). Inserting this result into (6.13) and using (2.9) gives

$$\begin{aligned} & bT\,\mathrm{Cov}(\widehat{\varphi}(u_1,\underline{s}_1),\widehat{\varphi}(u_2,\underline{s}_2)) \\ = \; & \frac{b}{T}\sum_{t_1,t_2=1}^{T} K_b\Big(\frac{t_1}{T}-u_1\Big)K_b\Big(\frac{t_2}{T}-u_2\Big)\mathrm{Cov}\big(g_{\underline{s}_1}(\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_{t_2}(\frac{t_2}{T}))\big) + \mathcal{O}(b). \end{aligned}$$

Therefore, the desired assertion follows if we can show that

$$\begin{aligned} & \frac{b}{T}\sum_{t_1,t_2=1}^{T} K_b\Big(\frac{t_1}{T}-u_1\Big)K_b\Big(\frac{t_2}{T}-u_2\Big)\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_{t_2}(\frac{t_2}{T}))\Big) \\ = \; & V((u_1,\underline{s}_1),(u_2,\underline{s}_2)) + o(1). \end{aligned}$$

From stationarity of $(\underline{\widetilde{X}}_t(u))_t$, we get

$$\frac{b}{T}\sum_{t_1,t_2=1}^{T} K_b\Big(\frac{t_1}{T}-u_1\Big)K_b\Big(\frac{t_2}{T}-u_2\Big)\mathrm{Cov}\big(g_{\underline{s}_1}(\underline{\widetilde{X}}_{t_1}(\frac{t_1}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_{t_2}(\frac{t_2}{T}))\big)$$

$$= \frac{1}{bT} \sum_{h=-(T-1)}^{T-1} \sum_{t=\max\{1,1-h\}}^{\min\{T,T-h\}} K\Big(\frac{\frac{t+h}{T}-u_1}{b}\Big) K\Big(\frac{\frac{t}{T}-u_2}{b}\Big) \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big). \tag{6.16}$$

In the following, rather than the last right-hand side, we consider

$$\frac{1}{bT} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} K\Big(\frac{\frac{t+h}{T}-u_1}{b}\Big) K\Big(\frac{\frac{t}{T}-u_2}{b}\Big) \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big) \tag{6.17}$$

with $\underline{\widetilde{X}}_v(z) = \underline{\widetilde{X}}_v(1)$ if $z \geq 1$ and $\underline{\widetilde{X}}_v(z) = \underline{\widetilde{X}}_v(0)$ if $z \leq 0$. This can be justified since the difference of (6.16) and (6.17) can be shown to be of order $\mathcal{O}((bT)^{-1})$ using the summability condition on $(l(j))_j$ stated in Assumption 2.1. In the next step, we replace the Riemann sum in (6.17) by its integral, i.e. we consider

$$\frac{1}{b} \sum_{h=-(T-1)}^{T-1} \int_0^1 K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) K\Big(\frac{y-u_2}{b}\Big) \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(y))\Big)\, dy. \tag{6.18}$$

To be allowed to consider (6.18) in the following, we have to show that the difference of (6.17) and (6.18) vanishes asymptotically. By standard arguments, its modulus can be bounded by

$$\frac{1}{b} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \Bigg| K\Big(\frac{\frac{t+h}{T}-u_1}{b}\Big) K\Big(\frac{\frac{t}{T}-u_2}{b}\Big) \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big)$$
$$- K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) K\Big(\frac{y-u_2}{b}\Big) \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(y))\Big)\Bigg| dy$$

and it remains to show that

$$\begin{aligned}
I &= \frac{1}{b} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} \Big| K\Big(\frac{\frac{t+h}{T}-u_1}{b}\Big) - K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) \Big| K\Big(\frac{\frac{t}{T}-u_2}{b}\Big) \\
&\quad \times \Big| \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big) \Big| dy, \\
II &= \frac{1}{b} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) \Big| K\Big(\frac{\frac{t}{T}-u_2}{b}\Big) - K\Big(\frac{y-u_2}{b}\Big) \Big| \\
&\quad \times \Big| \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big) \Big| dy, \\
III &= \frac{1}{b} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) K\Big(\frac{y-u_2}{b}\Big) \\
&\quad \times \Big| \mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(\frac{t+h}{T})) - g_{\underline{s}_1}(\underline{\widetilde{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\underline{\widetilde{X}}_0(\frac{t}{T}))\Big) \Big| dy, \\
IV &= \frac{1}{b} \sum_{h=-(T-1)}^{T-1} \sum_{t=1}^{T} \int_{\frac{t-1}{T}}^{\frac{t}{T}} K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big) K\Big(\frac{y-u_2}{b}\Big)
\end{aligned}$$

$$\times\Big|\mathrm{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(\frac{t}{T}))-g_{\underline{s}_2}(\widetilde{\underline{X}}_0(y))\Big)\Big|dy$$

vanish asymptotically. By Lipschitz continuity of the kernel K, $b^2T\to\infty$ and the absolute summability of $\mathrm{Cov}\big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h((t+h)/T)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(t/T))\big)$, cf. Lemma 2.1 (i), the first two terms I and II are of order $\mathcal{O}((b^2T)^{-1})$ and vanish asymptotically. Hence, it remains to consider III and IV, where we focus on III only, as the arguments are completely analogous for IV. In order to consider $\lim_{T\to\infty}|III|$, note that

$$|III|\le\sum_{h=-\infty}^{\infty}\mathbb{1}_{\{|h|\le T-1\}}|f_T(h)|$$

with an obvious notation for $f_T(h)$. We want to apply Lebesgue's dominated convergence theorem and derive the following bound for $|f_T(h)|$ using Lemma 2.1 (i) and the fact that K is bounded by a constant C:

$$\begin{aligned}
|f_T(h)| &\le \frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}} C\,K\Big(\frac{y-u_2}{b}\Big)\Big(\Big|\mathrm{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(\frac{t+h}{T})), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(\frac{t}{T}))\Big)\Big| \\
&\quad +\Big|\mathrm{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(\frac{t}{T}))\Big)\Big|\Big)dy \qquad (6.19)\\
&\le 2C\,2d(|\underline{s}_1|_1+|\underline{s}_2|_1)\,\big\|\underline{\varepsilon}_0\big\|_1\frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}K\Big(\frac{y-u_2}{b}\Big)\,dy\sum_{|j|\ge\lceil|h/2|\rceil}\frac{B}{l(j)}\\
&\le 4Cd\,(|\underline{s}_1|_1+|\underline{s}_2|_1)\,\big\|\underline{\varepsilon}_0\big\|_1\int_{-1}^{1}K(z)\,dz\sum_{|j|\ge\lceil|h/2|\rceil}\frac{B}{l(j)},
\end{aligned}$$

where the standard substitution $z=(y-u_2)/b$ was used. The bound on the right-hand side does not depend on T and is summable in h since

$$\sum_{h\in\mathbb{Z}}\sum_{|j|\ge\lceil|h/2|\rceil}\frac{B}{l(j)}\le\sum_{j\in\mathbb{Z}}(4|j|+1)\frac{B}{l(j)}<\infty.$$

Hence, we can apply Lebesgue's dominated convergence theorem to III and obtain

$$\begin{aligned}
\lim_{T\to\infty}\big|III\big| &\le \sum_{h=-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big)K\Big(\frac{y-u_2}{b}\Big)\\
&\quad\times\Big|\mathrm{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(\frac{t+h}{T}))-g_{\underline{s}_1}(\widetilde{\underline{X}}_h(y+\frac{h}{T})), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(\frac{t}{T}))\Big)\Big|dy\\
&\le \sum_{h=-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}C\,K\Big(\frac{y-u_2}{b}\Big)\\
&\quad\times 2E\Big|g_{\underline{s}_1}(\widetilde{\underline{X}}_h(\frac{t+h}{T}))-g_{\underline{s}_1}(\widetilde{\underline{X}}_h(y+\frac{h}{T}))\Big|dy, \qquad (6.20)
\end{aligned}$$

because $|\mathrm{Cov}(X,Y)|\le 2E|X|$ if $|Y|=1$. The expectation in (6.20) can be bounded using a Taylor expansion of $g_{\underline{s}_1}$ exactly as the one for the covariance term in (6.15). Thus, invoking Lemma 2.1 (iii), we get

$$E\Big|g_{\underline{s}_1}(\widetilde{\underline{X}}_h(\frac{t+h}{T}))-g_{\underline{s}_1}(\widetilde{\underline{X}}_h(y+\frac{h}{T}))\Big|\le|\underline{s}_1|_1\Big(Ld+\big\|\underline{\varepsilon}_0\big\|_1\cdot\sum_{j=-\infty}^{\infty}\frac{Bd}{l(j)}\Big)\cdot\Big|\frac{t}{T}-y\Big|, \qquad (6.21)$$

where $\underline{\xi}$ is between $\underline{\widetilde{X}}_h((t+h)/T)$ and $\underline{\widetilde{X}}_h(y+(h/T))$. With the bound from (6.21) the expression in (6.20) can be bounded by

$$\begin{aligned}
& 2C\,|\underline{s}_1|_1\Big(Ld+\big\|\underline{\varepsilon}_0\big\|_1\cdot\sum_{j=-\infty}^{\infty}\frac{Bd}{l(j)}\Big)\cdot\sum_{h=-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{b}\sum_{t=1}^{T}\int_{\frac{t-1}{T}}^{\frac{t}{T}}K\Big(\frac{y-u_2}{b}\Big)\Big|\frac{t}{T}-y\Big|\,dy \\
\leq\ & 2C\,|\underline{s}_1|_1\Big(Ld+\big\|\underline{\varepsilon}_0\big\|_1\cdot\sum_{j=-\infty}^{\infty}\frac{Bd}{l(j)}\Big)\cdot\sum_{h=-\infty}^{\infty}\lim_{T\to\infty}\frac{1}{T}\int_{-1}^{1}K(z)\,dz \\
=\ & 0.
\end{aligned}$$

Therefore, III vanishes asymptotically and, by the same arguments, it follows $IV=o(1)$. This allows us to consider (6.18) in the sequel.
We first consider the case $u_1\neq u_2$. It can be shown by standard arguments that

$$\lim_{T\to\infty}\int_{-1}^{1}K\Big(z+\frac{h}{bT}+\frac{u_2-u_1}{b}\Big)K(z)\,dz=0. \tag{6.22}$$

With the same calculations as in (6.19), we get for each $h\in\mathbb{Z}$ the bound

$$\begin{aligned}
& \frac{1}{b}\int_0^1 K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big)K\Big(\frac{y-u_2}{b}\Big)\Big|\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(y+\frac{h}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_0(y))\Big)\Big|\,dy \\
\leq\ & C\,2d(|\underline{s}_1|_1+|\underline{s}_2|_1)\,\big\|\underline{\varepsilon}_0\big\|_1\int_{-1}^{1}K(z)\,dz\sum_{j\geq\lceil|h/2|\rceil}\frac{B}{l(j)},
\end{aligned}$$

which is summable in h. This allows for the application of Lebesgue's dominated convergence theorem to (6.18) which, together with Lemma 2.1(i) and again $z=(y-u_2)/b$, yields

$$\begin{aligned}
& \lim_{T\to\infty}\Bigg|\frac{1}{b}\sum_{h=-(T-1)}^{T-1}\int_0^1 K\Big(\frac{y+\frac{h}{T}-u_1}{b}\Big)K\Big(\frac{y-u_2}{b}\Big)\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(y+\frac{h}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_0(y))\Big)\,dy\Bigg| \\
\leq\ & 2d(|\underline{s}_1|_1+|\underline{s}_2|_1)\,\big\|\underline{\varepsilon}_0\big\|_1\sum_{h=-\infty}^{\infty}\sum_{|j|\geq\lceil|h/2|\rceil}\frac{B}{l(j)}\lim_{T\to\infty}\int_{-1}^{1}K\Big(z+\frac{h}{bT}+\frac{u_2-u_1}{b}\Big)K(z)\,dz \\
=\ & 0,
\end{aligned}$$

due to (6.22), which proves the first part of assertion (6.12). Now, consider the case $u_1=u_2$. Then, by using the substitution $z=(y-u_1)/b$, (6.18) becomes

$$\begin{aligned}
& \sum_{h=-(T-1)}^{T-1}\int_{-1}^{1}K\Big(z+\frac{h}{bT}\Big)K(z)\,\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(u_1+bz+\frac{h}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_0(u_1+bz))\Big)\,dz \qquad (6.23) \\
=\ & \sum_{h=-(T-1)}^{T-1}\int_{-1}^{1}K^2(z)\,dz\cdot\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(u_1)),g_{\underline{s}_2}(\underline{\widetilde{X}}_0(u_1))\Big)+R1+R2+R3,
\end{aligned}$$

where

$$R1 = \sum_{h=-(T-1)}^{T-1}\int_{-1}^{1}\Big(K\Big(z+\frac{h}{bT}\Big)-K(z)\Big)K(z)\mathrm{Cov}\Big(g_{\underline{s}_1}(\underline{\widetilde{X}}_h(u_1+bz+\frac{h}{T})),g_{\underline{s}_2}(\underline{\widetilde{X}}_0(u_1+bz))\Big)\,dz,$$

$$
\begin{aligned}
R2 &= \sum_{h=-(T-1)}^{T-1} \int_{-1}^{1} K^2(z) \operatorname{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1 + bz + \frac{h}{T})) - g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_1 + bz)) \Big)\, dz, \\
R3 &= \sum_{h=-(T-1)}^{T-1} \int_{-1}^{1} K^2(z) \operatorname{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_1 + bz)) - g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_1)) \Big)\, dz.
\end{aligned}
$$

From boundedness and Lipschitz continuity of K, as well as Lemma 2.1(i), $R1$ can be bounded in absolute value by

$$
\mathcal{O}((bT)^{-1}) \sum_{h=-\infty}^{\infty} |h|\, 2d(|\underline{s}_1|_1 + |\underline{s}_2|_1) \left\|\underline{\varepsilon}_0\right\|_1 \sum_{|j| \geq \lceil |h/2| \rceil} \frac{B}{l(j)} = \mathcal{O}((bT)^{-1}).
$$

Therefore, $R1$ vanishes asymptotically. For $R2$ we have $|R2| \leq \sum_{h=-\infty}^{\infty} \mathbb{1}_{\{|h| \leq T-1\}} |\widetilde{f}_T(h)|$ with an obvious notation for $\widetilde{f}_T(h)$. Via Lemma 2.1(i) we have the bound

$$
|\widetilde{f}_T(h)| \;\leq\; \int_{-1}^{1} K^2(z)\, dz \cdot 4d(|\underline{s}_1|_1 + |\underline{s}_2|_1) \left\|\underline{\varepsilon}_0\right\|_1 \sum_{|j| \geq \lceil |h/2| \rceil} \frac{B}{l(j)},
$$

which is summable in h and not depending on T. Hence, we can apply Lebesgue's dominated convergence theorem which yields, again using the fact that $|\operatorname{Cov}(X,Y)| \leq 2E|X|$ if $|Y| = 1$,

$$
\lim_{T\to\infty} \big|\, R2 \,\big| \leq \sum_{h=-\infty}^{\infty} \lim_{T\to\infty} \int_{-1}^{1} K^2(z)\, 2E \Big| g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1 + bz + \frac{h}{T})) - g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)) \Big|\, dz = 0.
$$

With exactly the same arguments, $R3$ also converges to zero. Applying these results to (6.23), it holds

$$
\begin{aligned}
& bT \operatorname{Cov}(\widehat{\varphi}(u_1, \underline{s}_1), \widehat{\varphi}(u_1, \underline{s}_2)) \\
= & \sum_{h=-(T-1)}^{T-1} \int_{-1}^{1} K^2(z)\, dz \cdot \operatorname{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_1)) \Big) + o(1) \\
= & \int_{-1}^{1} K^2(z)\, dz \cdot \sum_{h=-\infty}^{\infty} \operatorname{Cov}\Big(g_{\underline{s}_1}(\widetilde{\underline{X}}_h(u_1)), g_{\underline{s}_2}(\widetilde{\underline{X}}_0(u_1)) \Big) + o(1) \qquad (6.24) \\
= & V((u_1, \underline{s}_1), (u_1, \underline{s}_2)) + o(1).
\end{aligned}
$$

The assertion for the real and imaginary parts follows immediately from $\Re x = (x + \overline{x})/2$ and $\Im x = (x - \overline{x})/(2i)$ and $\exp(ix) = \cos(x) + i \sin(x)$ which completes the proof of part (iii). ■

Proof of Theorem 3.1. For any $u \in (0,1)$ and $\underline{s} \in [-S,S]^d$, we can write

$$
\begin{aligned}
& \sqrt{bT}\, \left(\widehat{\varphi}(u, \underline{s}) - \varphi(u, \underline{s})\right) \\
= & \sqrt{bT}\, \left(\widehat{\varphi}(u, \underline{s}) - E[\widehat{\varphi}(u, \underline{s})]\right) + \sqrt{bT}\, \left(E[\widehat{\varphi}(u, \underline{s})] - \varphi(u, \underline{s})\right), \qquad (6.25)
\end{aligned}
$$

where the second summand on the right-hand side above converges to zero if $b^3 T \to 0$ holds and to $C\beta(u, \underline{s})$ if $b^5 T \to C^2$ holds, respectively, by Lemma 3.1(i,ii). Hence, it remains to show the

CLT in (3.6). By Cramér-Wold device, this is equivalent to show for all $\underline{c} \in \mathbb{R}^{2J}$ the corresponding CLT

$$Z_T := \sqrt{bT}\underline{c}' \left(\begin{pmatrix} \Re\left(\widehat{\varphi}(u_j, \underline{s}_j) - E(\widehat{\varphi}(u_j, \underline{s}_j))\right) \\ \Im\left(\widehat{\varphi}(u_j, \underline{s}_j) - E(\widehat{\varphi}(u_j, \underline{s}_j))\right) \end{pmatrix}, j = 1, \ldots, J \right) \xrightarrow{d} \mathcal{N}(0, \underline{c}'\mathbf{V}\underline{c}).$$

Now, we have to distinguish the two cases of a positive variance $\underline{c}'\mathbf{V}\underline{c}$ and a vanishing variance $\underline{c}'\mathbf{V}\underline{c} = 0$. In the latter case, we get $Z_T \xrightarrow{d} \delta_0$. Now, suppose $\underline{c}'\mathbf{V}\underline{c} > 0$ holds. Further, let $(\underline{X}_{t,T}^{(M)})$ be the truncated version of $(\underline{X}_{t,T})$, i.e.

$$\underline{X}_{t,T}^{(M)} = \underline{\mu}\Big(\frac{t}{T}\Big) + \sum_{|j|<M} A_{t,T}(j)\, \underline{\varepsilon}_{t-j},$$

and define

$$Z_T^{(M)} := \sqrt{bT}\underline{c}' \left(\begin{pmatrix} \Re\left(\widehat{\varphi}^{(M)}(u_j, \underline{s}_j) - E(\widehat{\varphi}^{(M)}(u_j, \underline{s}_j))\right) \\ \Im\left(\widehat{\varphi}^{(M)}(u_j, \underline{s}_j) - E(\widehat{\varphi}^{(M)}(u_j, \underline{s}_j))\right) \end{pmatrix}, j = 1, \ldots, J \right),$$

where $\widehat{\varphi}^{(M)}(u, \underline{s})$ is defined analogue to $\widehat{\varphi}(u, \underline{s})$, but with $\underline{X}_{t,T}$ replaced by $\underline{X}_{t,T}^{(M)}$. Further, we define the block covariance matrix $\mathbf{V}_M$ analogous to $\mathbf{V}$, but based on

$$V_M((u_1, \underline{s}_1), (u_2, \underline{s}_2)) = \int_{-1}^{1} K^2(x)\, dx \cdot \sum_{h=-2(M-1)}^{2(M-1)} \mathrm{Cov}\big(\exp\left(i\, \langle \underline{s}_1\, ,\, \widetilde{\underline{X}}_0^{(M)}(u_1)\rangle\right), \exp\left(i\, \langle \underline{s}_2\, ,\, \widetilde{\underline{X}}_h^{(M)}(u_2)\rangle\right)\big)$$

instead of $V((u_1, \underline{s}_1), (u_2, \underline{s}_2))$. Now, we can make use of Proposition 6.3.9 in Brockwell and Davis (1991) and we have to show

$$\begin{aligned} &(a) \quad \exists M_0 \in \mathbb{N}\ \forall M \geq M_0 :\ Z_T^{(M)} \xrightarrow{\mathcal{D}} \mathcal{N}(0, \underline{c}'\mathbf{V}_M\underline{c}) \text{ as } T \to \infty, \\ &(b) \quad \underline{c}'\mathbf{V}_M\underline{c} \longrightarrow \underline{c}'\mathbf{V}\underline{c} \text{ as } M \to \infty, \\ &(c) \quad \forall\ \delta > 0 :\ \lim_{M\to\infty} \limsup_{T\to\infty} P(|Z_T - Z_T^{(M)}| \geq \delta) = 0. \end{aligned}$$

First, we get immediately from Lemma 2.1 (i) that $\underline{c}'\mathbf{V}_M\underline{c} \to \underline{c}'\mathbf{V}\underline{c} < \infty$ as $M \to \infty$, which proves (b). Now turn to part (a). By using completely the same arguments as used to compute the variance in Lemma 3.1(iii), we can show that

$$\mathrm{Var}(Z_T^{(M)}) = \underline{c}'\mathbf{V}_M\underline{c} + o(1).$$

From $\underline{c}'\mathbf{V}\underline{c} > 0$ and as $\mathbf{V}_M \to \mathbf{V}$ for $M \to \infty$, we get also $\underline{c}'\mathbf{V}_M\underline{c} > 0$ for all $M \geq M_0$ and M_0 sufficient large. Hence, it suffices to show that

$$\frac{Z_T^{(M)}}{\sqrt{\mathrm{Var}(Z_T^{(M)})}} \xrightarrow{d} \mathcal{N}(0, 1) \text{ as } T \to \infty. \tag{6.26}$$

To prove this, we write

$$Y_{t,T}^{(M)} = \sqrt{\frac{b}{T}}\underline{c}' \left(K_b\left(\frac{t}{T} - u_j\right) \begin{pmatrix} \cos(\langle \underline{s}_j, \underline{X}_{t,T}^{(M)}\rangle) - E[\cos(\langle \underline{s}_j, \underline{X}_{t,T}^{(M)}\rangle)] \\ \sin(\langle \underline{s}_j, \underline{X}_{t,T}^{(M)}\rangle) - E[\sin(\langle \underline{s}_j, \underline{X}_{t,T}^{(M)}\rangle)] \end{pmatrix}, j = 1, \ldots, J \right),$$

where we suppress the dependence on u_j and $\underline{s}_j$. Note that all summands in $Y_{t,T}^{(M)}$ with $|(t/T - u)/b| > 1$ are zero since the kernel K has compact support $[-1,1]$. Consequently, as we consider $u_1, \ldots, u_J$, at most $d_T = J(2\lfloor bT \rfloor + 1)$ (subsequent) of the summands $Y_{t,T}^{(M)}$ above fulfill $|(t/T - u_j)/b| \leq 1$ for at least one j. Let $Y_{t_1,T}^{(M)}, \ldots, Y_{t_{d_T},T}^{(M)}$ denote these non-vanishing summands such that we can write

$$Z_T^{(M)} = \sum_{r=1}^{d_T} Y_{t_r,T}^{(M)}.$$

Note that $(Y_{t_r,T}^{(M)}, r = 1, \ldots, d_T)$ forms a triangular array of centered $(2(M-1))$-dependent random variables such that we can use the CLT in Theorem 2.1 in Romano and Wolf (2000), which is tailor-made for m-dependent random variables. In the following, we adapt their notation and we have to check their Conditions (1) - (6). We refer to Romano and Wolf (2000) for details. Since

$$\begin{aligned} & E[|Y_{t_r,T}^{(M)}|^{2+\delta}] \\ = \; & E\left[\left|\sqrt{\frac{b}{T}}\underline{c}'\left(K_b\left(\frac{t_r}{T} - u_j\right)\begin{pmatrix} \cos(\langle \underline{s}_j, \underline{X}_{t_r,T}^{(M)}\rangle) - E[\cos(\langle \underline{s}_j, \underline{X}_{t_r,T}^{(M)}\rangle)] \\ \sin(\langle \underline{s}_j, \underline{X}_{t_r,T}^{(M)}\rangle) - E[\sin(\langle \underline{s}_j, \underline{X}_{t_r,T}^{(M)}\rangle)] \end{pmatrix}, j = 1, \ldots, J\right)\right|^{2+\delta}\right] \\ \leq \; & C_\Delta \, (bT)^{-(1+\frac{\delta}{2})} \end{aligned}$$

for some finite constant C_Δ and any $\delta > 0$, we get that Condition (1) holds with $\Delta_T = C_\Delta \, (bT)^{-(1+\frac{\delta}{2})}$. As all $|\mathrm{Cov}\left(\cos(\langle \underline{s}_j, \underline{X}_{t_1,T}^{(M)}\rangle), \cos(\langle \underline{s}_j, \underline{X}_{t_2,T}^{(M)}\rangle)\right)|$, $|\mathrm{Cov}\left(\sin(\langle \underline{s}_j, \underline{X}_{t_1,T}^{(M)}\rangle), \sin(\langle \underline{s}_j, \underline{X}_{t_2,T}^{(M)}\rangle)\right)|$ as well as $|\mathrm{Cov}\left(\sin(\langle \underline{s}_j, \underline{X}_{t_1,T}^{(M)}\rangle), \cos(\langle \underline{s}_j, \underline{X}_{t_2,T}^{(M)}\rangle)\right)|$ are bounded by 2, we get $|\mathrm{Cov}\left(Y_{t_{r_1}}^{(M)}, Y_{t_{r_2}}^{(M)}\right)| = O((bT)^{-1})$. Hence, with $\gamma = 0$ in their notation, we have for all a and all $k \geq 1$ that

$$\frac{1}{k}\mathrm{Var}\left(\sum_{r=a}^{a+k-1} Y_{t_r,T}^{(M)}\right) = \sum_{h=\max(-(k-1),-2(M-1))}^{\min(k-1,2(M-1))} \frac{1}{k} \sum_{s=\max(1,1-h)}^{\min(k,k-h)} \mathrm{Cov}(Y_{t_{s+h+a-1},T}^{(M)}, Y_{t_{s+a-1},T}^{(M)}) \leq C_K \cdot (bT)^{-1}$$

for some suitable positive and finite constant C_K. This means that Condition (2) holds with $K_T = C_K \cdot (bT)^{-1}$. Further, since $\underline{c}'\mathbf{V}_M\underline{c} > 0$ for all $M \geq M_0$ and M_0 sufficiently large, and as $\mathrm{Var}\left(\sum_{r=1}^{d_T} Y_{t_r,T}^{(M)}\right) = \underline{c}'\mathbf{V}_M\underline{c} + o(1)$, we obtain for sufficiently large T, that there exists a strictly positive and finite constant C_L such that

$$\frac{1}{d_T}\mathrm{Var}\left(\sum_{r=1}^{d_T} Y_{t_r,T}^{(M)}\right) \geq C_L \cdot (bT)^{-1}.$$

This means that Condition (3) holds with $L_T = C_L \cdot (bT)^{-1}$. Altogether, this leads to Conditions (4) and (5) being satisfied, i.e.

$$(4) \quad \frac{K_T}{L_T} = \frac{C_K(bT)^{-1}}{C_L(bT)^{-1}} = \mathcal{O}(1) \quad \text{and} \quad (5) \quad \frac{\Delta_T}{L_T^{1+\delta/2}} = \frac{C_\Delta \, (bT)^{-(1+\frac{\delta}{2})})}{(C_L(bT)^{-1})^{1+\delta/2}} = \mathcal{O}(1).$$

As their Condition (6) is trivially fulfilled as M is fixed here, this proves the CLT in (6.26) and completes part (a). Finally, to show (c), by Markov inequality, it suffices to consider $E\left(|Z_T - Z_T^{(M)}|^2\right)$

in more detail. To avoid lengthy notation, we treat only the case of $J = 1$ and consider only the real part. Similar to (6.24) in the proof of Lemma 3.1(iii), we get

$$\begin{aligned}
& bT \operatorname{Var}\left(\Re\widehat{\varphi}(u,\underline{s}) - \Re\widehat{\varphi}^{(M)}(u,\underline{s})\right) \\
= & \sum_{h=-(T-1)}^{T-1} \int_{-1}^{1} K^2(z)\,dz \operatorname{Cov}\left(\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u)), \Re g_{\underline{s}}(\underline{\widetilde{X}}_0(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(M)}(u))\right) + o(1) \\
=: & \; I_{T,M} + o(1)
\end{aligned}$$

as $T \to \infty$, where $\underline{\widetilde{X}}_t^{(M)}(u) = \underline{\mu}(u) + \sum_{|j|<M} A(u,j)\,\underline{\varepsilon}_{t-j}$ is the truncated version of $\widetilde{X}_t(u)$ and $g_{\underline{s}}(\underline{x}) := \exp(i\langle \underline{s}, \underline{x}\rangle)$. Hence, it suffices to prove $\lim_{M\to\infty} \limsup_{T\to\infty} |I_{T,M}| = 0$ by making use of Lebesgue's Theorem. By using the same approach as in the proof of Lemma 3.1(iii) to show the finiteness of the variance $\mathbf{V}$, the modulus of the covariances in $I_{T,M}$ above can be bounded by

$$\begin{aligned}
& \Big|\operatorname{Cov}\Big(\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u)) - \left\{\Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(v)}(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M,v)}(u))\right\}, \\
& \qquad \Re g_{\underline{s}}(\underline{\widetilde{X}}_0(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(M)}(u))\Big)\Big| \\
& + \Big|\operatorname{Cov}\Big(\Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(v)}(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M,v)}(u)), \\
& \qquad g(\underline{\widetilde{X}}_0(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(M)}(u)) - \left\{\Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(v)}(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(M,v)}(u))\right\}\Big)\Big|,
\end{aligned}$$

where $\underline{\widetilde{X}}_t^{(M,v)}(u) = \underline{\mu}(u) + \sum_{|j|<\min(M,v)} A(u,j)\,\underline{\varepsilon}_{t-j}$ and $v = \lceil |h/2| \rceil$. The first summand above can be bounded by

$$\begin{aligned}
& 4E\left(\left|\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u)) - \left\{\Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(v)}(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M,v)}(u))\right\}\right|\right) \\
\le \; & 4\left\{E\left(\left|\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(v)}(u))\right|\right) + E\left(\left|\Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M,v)}(u))\right|\right)\right\} \\
\le \; & 4\,|\underline{s}|_1 \|\underline{\varepsilon}_0\|_1 \left\{\sum_{|j|\ge v} |A(u,j)|_1 + \sum_{M>|j|\ge \min(v,M)} |A(u,j)|_1\right\} \\
\le \; & 8\,d\,|\underline{s}|_1\,\|\underline{\varepsilon}_0\|_1 \sum_{|j|\ge v} \frac{B}{l(j)}
\end{aligned}$$

and the second summand can be treated completely analogue. Altogether, the summands can be bounded by summable coefficients not depending on M, which allows to bound $\lim_{M\to\infty} \limsup_{T\to\infty} |I_{T,M}|$ applying Lebesgue's theorem as follows

$$\begin{aligned}
& \lim_{M\to\infty} \limsup_{T\to\infty} |I_{T,M}| \\
\le \; & \lim_{M\to\infty} \lim_{T\to\infty} \sum_{h=-\infty}^{\infty} \int_{-1}^{1} K^2(z)\,dz \Big|\operatorname{Cov}\Big(\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u)), \\
& \Re g_{\underline{s}}(\underline{\widetilde{X}}_0(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_0^{(M)}(u))\Big)\Big|
\end{aligned}$$

$$
\begin{aligned}
&\leq 4 \sum_{h=-\infty}^{\infty} \int_{-1}^{1} K^2(z)\, dz \left\{ \lim_{M\to\infty} E\left(|\Re g_{\underline{s}}(\underline{\widetilde{X}}_h(u)) - \Re g_{\underline{s}}(\underline{\widetilde{X}}_h^{(M)}(u))| \right) \right\} \\
&\leq 4d|\underline{s}|_1 \|\underline{\varepsilon}_0\|_1 \sum_{h=-\infty}^{\infty} \int_{-1}^{1} K^2(z)\, dz \left\{ \lim_{M\to\infty} \sum_{|j|\geq M} \frac{B}{l(j)} \right\} \\
&= 0,
\end{aligned}
$$

which proves part (c) and concludes this proof. ■

Proof of Lemma 3.2. Applying Lemma 2.1(ii) and (iii) we obtain

$$
\begin{aligned}
&E \sup_{s\in[-S,S]^d} \sqrt{bT} \Bigg| \widehat{\varphi}(u,\underline{s}) - E\widehat{\varphi}(u,\underline{s})) \\
&\qquad - \frac{1}{T}\sum_{t=1}^{T} K_b\left(\frac{t}{T}-u\right) \left[\exp(i\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle) - E\exp(i\langle \underline{s}, \underline{\widetilde{X}}_t(u)\rangle) \right] \Bigg| \\
&\leq \frac{2S}{\sqrt{bT}} \sum_{t=1}^{T} K\left(\frac{t/T-u}{b}\right) \left\{ \left\| \underline{X}_{t,T} - \underline{\widetilde{X}}_t(t/T) \right\|_1 + \left\| \underline{\widetilde{X}}_t(t/T) - \underline{\widetilde{X}}_t(u) \right\|_1 \right\} \\
&\leq \mathcal{O}\left(\sqrt{\frac{b}{T}}\right) + \mathcal{O}(\sqrt{b^3T}),
\end{aligned}
$$

which vanishes asymptotically. ■

Proof of Lemma 3.3. We restrict ourselves to the first assertion since the second one can be derived in complete analogy. For notational simplicity, we define $\bar{g}_{\underline{s}}(\underline{X}) = \cos(\underline{s}'\underline{X}) - E\cos(\underline{s}'\underline{X})$ for any $\underline{s} \in \mathbb{R}^d$ and any $\mathbb{R}^d$-valued random variable $\underline{X}$. Recall that the number of nonvanishing summands is bounded by $\lfloor 2bT \rfloor + 1$ (see end of Section 2) and that these summands are subsequent. Let t_1 denote the smallest of these indices. Inspired by Arcones, Yu (1994, Section 2), we divide this set of indices into blocks H_t, T_t (both of equal length ς_T) for $t = 1, \ldots, \mu_T$ and a remainder term R, where

$$
\begin{aligned}
H_t &:= \left\{ i \mid 2(t-1)\varsigma_T + t_1 \leq i < (2t-1)\varsigma_T + t_1 \right\}, \\
T_t &:= \left\{ i \mid (2t-1)\varsigma_T + t_1 \leq i < 2t\varsigma_T + t_1 \right\}, \\
R &:= \left\{ i \mid 2\mu_T\varsigma_T + t_1 \leq i < t_1 + \lceil 2bT \rceil \right\}.
\end{aligned}
$$

Here ς_T and μ_T have the following form for all $T \in \mathbb{N}$:

$$
\varsigma_T := \left\lfloor (bT)^{\frac{1}{2m}} \right\rfloor \qquad \text{and} \qquad \mu_T := \left\lfloor \frac{bT}{\varsigma_T} \right\rfloor. \tag{6.27}
$$

Further, let $\rho(\underline{s}_1, \underline{s}_2) = |\underline{s}_1 - \underline{s}_2|_1$. This gives

$$
\begin{aligned}
&\lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|(bT)^{-\frac{1}{2}}\sum_{t=1}^{bT} K\left(\frac{t/T-u}{b}\right)\left[\bar{g}_{\underline{s}_1}(\underline{\widetilde{X}_t}(u)) - \bar{g}_{\underline{s}_2}(\underline{\widetilde{X}_t}(u))\right]\right| > \lambda\right)\\
&\le \lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t} K\left(\frac{i/T-u}{b}\right)\left[\bar{g}_{\underline{s}_1}(\underline{\widetilde{X}_i}(u)) - \bar{g}_{\underline{s}_2}(\underline{\widetilde{X}_i}(u))\right]\right| > \frac{\lambda}{3}\right)\\
&\quad+ \lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in T_t} K\left(\frac{i/T-u}{b}\right)\left[\bar{g}_{\underline{s}_1}(\underline{\widetilde{X}_i}(u)) - \bar{g}_{\underline{s}_2}(\underline{\widetilde{X}_i}(u))\right]\right| > \frac{\lambda}{3}\right)\\
&\quad+ \lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\left|(bT)^{-\frac{1}{2}}\sum_{i\in R} K\left(\frac{i/T-u}{b}\right)\left[\bar{g}_{\underline{s}_1}(\underline{\widetilde{X}_i}(u)) - \bar{g}_{\underline{s}_2}(\underline{\widetilde{X}_i}(u))\right]\right| > \frac{\lambda}{3}\right)\\
&=: \underline{\mathrm{I}} + \underline{\mathrm{II}} + \underline{\mathrm{III}}.
\end{aligned}
\tag{6.28}
$$

As the second summand can be treated similarly to the first one, we focus on $\underline{\mathrm{I}}$ and $\underline{\mathrm{III}}$ in the sequel. For the third summand $\underline{\mathrm{III}}$ of (6.28), we have

$$
\begin{aligned}
\underline{\mathrm{III}} &\le \lim_{r\to 0}\limsup_{T\to\infty} P\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\|K\|_\infty (bT)^{-\frac{1}{2}}\sum_{i\in R}\left|\bar{g}_{\underline{s}_1}(\underline{\widetilde{X}}_i(u)) - \bar{g}_{\underline{s}_2}(\underline{\widetilde{X}}_i(u))\right| > \frac{\lambda}{3}\right)\\
&\le \limsup_{T\to\infty} P\left(C\,(bT)^{-1/2}\varsigma_T > \frac{\lambda}{3}\right)\\
&= 0,
\end{aligned}
$$

as it follows from $m > 1$ that $\varsigma_T = o((bT)^{1/2})$. In the next step we approximate the statistics under consideration by a statistic based on variables that are constructed such that the involved random variables with indices in different blocks $H_1, \ldots, H_{\mu_T}$ are independent. To this end, we use again the truncated version $\underline{\widetilde{X}}_i^{(M)}(u) = \underline{\mu}(u) + \sum_{|k|<M} A(u,k)\,\underline{\varepsilon}_{i-k}$ for $M = \lceil \varsigma_T/2 \rceil$. Using a first order

Taylor expansion of $\bar{g}_{\underline{s}}$, the first summand of (6.28) can be rewritten as follows

$$
\begin{aligned}
\underline{\mathrm{I}} &= \lim_{r\to 0}\limsup_{T\to\infty} P\Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\Bigg|(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t}K\Big(\frac{i/T-u}{b}\Big)\Big[\bar{g}_{\underline{s}_1}\big(\underline{\widetilde{X}}_i^{(M)}(u)\big)-\bar{g}_{\underline{s}_2}\big(\underline{\widetilde{X}}_i^{(M)}(u)\big)\\
&\qquad\qquad + \nabla\Big(\bar{g}_{\underline{s}_1}\big(\underline{Y}_i^*\big)-\bar{g}_{\underline{s}_2}\big(\underline{Y}_i^*\big)\Big)\big(\underline{\widetilde{X}}_i(u)-\underline{\widetilde{X}}_i^{(M)}(u)\big)\Big]\Bigg|>\frac{\lambda}{3}\Bigg)\\
&\le \lim_{r\to 0}\limsup_{T\to\infty} P\Bigg(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)<r}}\Bigg|(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t}K\Big(\frac{i/T-u}{b}\Big)\Big[\bar{g}_{\underline{s}_1}\big(\underline{\widetilde{X}}_i^{(M)}(u)\big)-\bar{g}_{\underline{s}_2}\big(\underline{\widetilde{X}}_i^{(M)}(u)\big)\Big]\Bigg|>\frac{\lambda}{6}\Bigg)\\
&\quad + \limsup_{T\to\infty} P\Bigg(2d\,S\,|K|_\infty(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t}|\underline{\widetilde{X}}_i(u)-\underline{\widetilde{X}}_i^{(M)}(u)|_1>\frac{\lambda}{6}\Bigg)\\
&=: \lim_{r\to 0}\limsup_{T\to\infty}\underline{\mathrm{I}}\mathrm{a} + \limsup_{T\to\infty}\underline{\mathrm{I}}\mathrm{b}
\end{aligned}
\tag{6.29}
$$

for some $\underline{Y}_i^*$ between $\underline{\widetilde{X}}_i(u)$ and $\underline{\widetilde{X}}_i^{(M)}(u)$. Now $\underline{\mathrm{I}}\mathrm{b}$ goes to 0 with $T\to\infty$ since Markov's inequality gives

$$
\begin{aligned}
\limsup_{T\to\infty}\underline{\mathrm{I}}\mathrm{b} &\le \limsup_{T\to\infty}\frac{12}{\lambda}\,d\,S\,|K|_\infty\,(bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t}\|\underline{\widetilde{X}}_i(u)-\underline{\widetilde{X}}_i^{(M)}(u)\|_1\\
&= \limsup_{T\to\infty} C\,(bT)^{-\frac{1}{2}}\,\mu_T\sum_{i\in H_1}\sum_{|k|\ge M}\frac{1}{l(k)}\\
&\le \limsup_{T\to\infty} C\,(bT)^{-\frac{1}{2}}\mu_T\varsigma_T^{-m+1}\sum_{|k|\ge\lceil\frac{\varsigma_T}{2}\rceil}\frac{k^m}{l(k)}\\
&= 0.
\end{aligned}
$$

Hence, it remains to consider term $\underline{\mathrm{I}}\mathrm{a}$, which is more involved. Note that the random variables $(\underline{\widetilde{X}}_i^{(M)}(u)))_{i\in H_{t_1}}$ and $(\underline{\widetilde{X}}_i^{(M)}(u)))_{i\in H_{t_2}}$ are independent for $t_1\neq t_2$, which will allow us to apply standard empirical process theory in the sequel. We adapt the arguments of Arcones, Yu (1994) and introduce some notation first. Let $\nu_T(\underline{s})$ and $\nu_T(\underline{s}_1,\underline{s}_2)$ with $\underline{s},\underline{s}_1,\underline{s}_2\in[-S,S]^d$ be defined as

$$
\nu_T(\underline{s}) := (bT)^{-\frac{1}{2}}\sum_{t=1}^{\mu_T}\sum_{i\in H_t}K\Big(\frac{i/T-u}{b}\Big)\bar{g}_{\underline{s}}\big(\underline{\widetilde{X}}_i^{(M)}(u)\big)\quad\text{and}\quad \nu_T(\underline{s}_1,\underline{s}_2):=\nu_T(\underline{s}_1)-\nu_T(\underline{s}_2).
$$

Now, we use a classical chaining argument inspired by Arcones, Yu (1994). To this end, we introduce a decreasing sequence $r_k := r2^{-k},\ k=0,\dots,k_T$, for some r specified below and such that

$$
(bT)^{3(1-m)/(4m)}\le r_{k_T}\le (bT)^{-1/(m\delta)}. \tag{6.30}
$$

Note that for our choice of m such a k_T exists since for large T

$$
(bT)^{3(1-m)/(4m)}\le (bT)^{-1/(m\delta)}\quad\Longleftrightarrow\quad m\ge 1+\frac{4}{3\delta}.
$$

Let $\mathcal{F}_k \subset [-S,S]^d$ for $k \in \mathbb{N}_0$ be a collection of indices with

$$\#\mathcal{F}_k = D_k = D(r_k, [-S,S]^d, \rho) \qquad \text{and} \qquad \sup_{\underline{s}_1 \in [-S,S]^d} \min_{\underline{s}_2 \in \mathcal{F}_k} \rho(\underline{s}_1, \underline{s}_2) < r_k,$$

where $D(u, [-S,S]^d, \rho) = \max\{\#\mathcal{T}_0 \mid \mathcal{T}_0 \subseteq [-S,S]^d,\ \rho(\underline{s}_1, \underline{s}_2) > u\ \forall\, \underline{s}_1 \neq \underline{s}_2 \in [-S,S]^d\}$ denotes the usual packing number defined e.g. in van der Vaart and Wellner (2000, Definition 2.2.3). Obviously, for any $u > 0$

$$D(u, [-S,S]^d, \rho) \leq \left(\frac{2Sd}{u} + 1\right)^d$$

and hence, $D_k = \mathcal{O}(r_k^{-d})$. Then there exists maps $\pi_k \colon [-S,S]^d \to \mathcal{F}_k$ such that

$$|\underline{s} - \pi_k \underline{s}|_1 \leq r_k \quad \forall \underline{s} \in [-S,S]^d.$$

Thereby, we have

$$\begin{aligned}
&\sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} |\nu_T(\underline{s}_1, \underline{s}_2)| \\
&\leq \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \{|\nu_T(\underline{s}_1) - \nu_T(\underline{s}_2) - \nu_T(\pi_{k_T}\underline{s}_1) + \nu_T(\pi_{k_T}\underline{s}_2)| + |\nu_T(\pi_0\underline{s}_1) - \nu_T(\pi_0\underline{s}_2)|\} \\
&\quad + \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} \left| \sum_{k=1}^{k_T} \nu_T(\pi_k\underline{s}_1) - \nu_T(\pi_{k-1}\underline{s}_1) - \nu_T(\pi_k\underline{s}_2) + \nu_T(\pi_{k-1}\underline{s}_2) \right| \\
&\leq 2 \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} |\nu_T(\underline{s}_1, \underline{s}_2)| + \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 2r}} |\nu_T(\underline{s}_1, \underline{s}_2)| + 2 \sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3r_k}} |\nu_T(\underline{s}_1, \underline{s}_2)|.
\end{aligned}$$

Moreover, we define for $k \in \mathbb{N}$ and for some C^* specified below

$$\lambda_k := r_k^{\frac{1}{6}} \vee \left(C^* \, r_k^{\frac{1}{3}} (\log D_k)^{\frac{1}{2}} \right) \tag{6.31}$$

and let r be small enough to ensure $2\sum_{k=1}^{\infty} \lambda_k \leq \frac{\lambda}{18}$. Note that $D_k = \mathcal{O}(r_k^{-d})$ guaranties summability of $(\lambda_k)_k$. Hence, we have for $\underline{\mathrm{I}}$a in (6.29)

$$\begin{aligned}
\underline{\mathrm{I}}\mathrm{a} &= P\left\{ \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) < r}} |\nu_T(\underline{s}_1, \underline{s}_2)| > \frac{\lambda}{6} \right\} \\
&\leq P\left\{ 2 \sup_{\substack{\underline{s}_1, \underline{s}_2 \in [-S,S]^d \\ \rho(\underline{s}_1, \underline{s}_2) \leq r_{k_T}}} |\nu_T(\underline{s}_1, \underline{s}_2)| > \frac{\lambda}{18} \right\} + P\left\{ 2 \sum_{k=1}^{k_T} \sup_{\substack{\underline{s}_1 \in \mathcal{F}_k, \underline{s}_2 \in \mathcal{F}_{k-1} \\ \rho(\underline{s}_1, \underline{s}_2) \leq 3r_k}} |\nu_T(\underline{s}_1, \underline{s}_2)| > 2 \sum_{k=1}^{k_T} \lambda_k \right\} \qquad (6.32) \\
&\quad + P\left\{ \sup_{\substack{\underline{s}_1, \underline{s}_2 \in \mathcal{F}_0 \\ \rho(\underline{s}_1, \underline{s}_2) \leq 2r}} |\nu_T(\underline{s}_1, \underline{s}_2)| > \frac{\lambda}{18} \right\} \\
&=: \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

Hence, it remains to show asymptotic negligibility of I, II, and III in order to prove the lemma. We start with the second summand an apply Bernstein's inequality for sums of independent random variables to the outer sum in the definition of ν_T:

$$\begin{aligned}
\mathrm{II} &\leq \sum_{k=1}^{k_T} P\Bigg(\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r_k}} |\nu_T(\underline{s}_1,\underline{s}_2)| > \lambda_k \Bigg) \\
&\leq \sum_{k=1}^{k_T} \sum_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r_k}} P\Big(\Big|(bT)^{\frac{1}{2}}\nu_T(\underline{s}_1,\underline{s}_2)\Big| > (bT)^{\frac{1}{2}}\lambda_k \Big) \\
&\leq 2\sum_{k=1}^{k_T} \exp\left\{ 2\log D_k - \frac{1}{2}\frac{bT\lambda_k^2}{A_{II,k} + 4|K|_\infty \varsigma_T \frac{(bT)^{\frac{1}{2}}\lambda_k}{3}} \right\}.
\end{aligned} \tag{6.33}$$

Here,

$$A_{II,k} \geq \sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r_k}} \mathrm{Var}\Big((bT)^{\frac{1}{2}}\nu_T(\underline{s}_1,\underline{s}_2) \Big).$$

For the variance term, we obtain

$$\begin{aligned}
&bT\,\mathrm{Var}\Big(\nu_T(\underline{s}_1,\underline{s}_2)\Big) \\
&\leq \mathrm{Var}\Bigg(\sum_{t=1}^{\mu_T}\sum_{i\in H_t} K\Big(\frac{i/T-u}{b}\Big)\big[\bar{g}_{\underline{s}_1}\big(\widetilde{\underline{X}}_i^{(M)}(u)\big) - \bar{g}_{\underline{s}_2}\big(\widetilde{\underline{X}}_i^{(M)}(u)\big)\big] \Bigg) \\
&\leq C\,\mu_T \sum_{i_1,i_2\in H_1} \mathrm{Cov}\Big[\Big(\bar{g}_{\underline{s}_1}\big(\widetilde{\underline{X}}_{i_1}^{(M)}(u)\big) - \bar{g}_{\underline{s}_2}\big(\widetilde{\underline{X}}_{i_1}^{(M)}(u)\big)\Big),\ \Big(\bar{g}_{\underline{s}_1}\big(\widetilde{\underline{X}}_{i_2}^{(M)}(u)\big) - \bar{g}_{\underline{s}_2}\big(\widetilde{\underline{X}}_{i_2}^{(M)}(u)\big)\Big)\Big].
\end{aligned} \tag{6.34}$$

In analogy to the proof of Lemma 2.1(i) we can bound the latter term by

$$\begin{aligned}
bT\,\mathrm{Var}\Big(\nu_T(\underline{s}_1,\underline{s}_2)\Big) &\leq C\,\mu_T \sum_{i_1,i_2\in H_1} \min\left\{ |\underline{s}_1-\underline{s}_2|_1\,,\ \sum_{j>\lfloor|i_1-i_2|/2\rfloor} \frac{1}{l(j)} \right\} \\
&\leq C\,\mu_T \sum_{t=0}^{\varsigma_T-1} (\varsigma_T - t) \min\left\{ |\underline{s}_1-\underline{s}_2|_1\,,\ \sum_{j>\lfloor t/2\rfloor} \frac{1}{l(j)} \right\} \\
&\leq C\,bT\Bigg[\sum_{t=0}^{R_0-1} |\underline{s}_1-\underline{s}_2|_1 + \sum_{t=R_0}^{\varsigma_T-1} t^{-3} \Bigg] \\
&\leq C\,bT\Big[R_0\,|\underline{s}_1-\underline{s}_2|_1 + R_0^{-2} \Big]
\end{aligned}$$

for any $R_0 \geq 2$. This holds for $R_0 := \Big\lfloor |\underline{s}_1-\underline{s}_2|_1^{-1/3} \Big\rfloor$, if r is sufficiently small. Then we have

$$\sup_{\substack{\underline{s}_1\in\mathcal{F}_k,\underline{s}_2\in\mathcal{F}_{k-1}\\ \rho(\underline{s}_1,\underline{s}_2)\leq 3r_k}} \mathrm{Var}\Big((bT)^{\frac{1}{2}}\nu_T(\underline{s}_1,\underline{s}_2) \Big) \leq C\,bT\,r_k^{2/3} =:\ A_{II,k}. \tag{6.35}$$

Hence, we obtain from (6.33) and (6.35)

$$\begin{aligned}
\mathrm{II} &\leq 2\sum_{k=1}^{k_T}\exp\left\{2\log D_k - \frac{1}{2}\frac{bT\lambda_k^2}{A_{II,k}+4\|K\|_\infty \varsigma_T \frac{(bT)^{\frac{1}{2}}\lambda_k}{3}}\right\}\\
&\leq 2\sum_{k=1}^{k_T}\exp\left\{2\log D_k - C\frac{\lambda_k^2}{r_k^{2/3}+r_{k_T}^{2/3}\lambda_k}\right\}\\
&\leq 2\sum_{k=1}^{k_T}\exp\left\{2\log D_k - C\frac{\lambda_k^2}{r_k^{2/3}+r_k^{2/3}\lambda}\right\}\\
&\leq 2\sum_{k=1}^{k_T}\exp\left\{2\log D_k - \bar{C}\lambda_k^2 r_k^{-2/3}\right\}
\end{aligned}$$

for some $\bar{C}\in(0,\infty)$. Setting $C^* = (4/\bar{C})^{1/2}$ in definition (6.31) of λ_k yields

$$\mathrm{II} \leq 2\sum_{k=1}^{k_T}\exp\left\{-\frac{\bar{C}}{2}\lambda_k^2 r_k^{-2/3}\right\} \leq 2\sum_{k=1}^{\infty}\exp\left\{-\frac{\bar{C}}{2}r_k^{-1/3}\right\} \xrightarrow[r\to 0]{} 0.$$

Analogously, we have for summand III in (6.32)

$$P\left\{\sup_{\substack{\underline{s}_1,\underline{s}_2\in\mathcal{F}_0\\ \rho(\underline{s}_1,\underline{s}_2)\leq 2r}}|\nu_T(\underline{s}_1,\underline{s}_2)|>\lambda\right\} \leq 2\exp\left\{2\log D_0 - \frac{1}{2}\frac{bT\lambda^2}{A_{III,k}+4\|K\|_\infty\varsigma_T\frac{(bT)^{\frac{1}{2}}\lambda}{3}}\right\},$$

where $A_{III,k} = C\,bT\,r^{2/3}$. As before, we get asymptotic negligibility as $r\to 0$.

Finally, we look at the first summand in (6.32). Applying Markov's inequality, it suffices to verify

$$\lim_{r\to 0}\limsup_{T\to\infty} E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}|\nu_T(\underline{s}_1,\underline{s}_2)|\right) = 0, \tag{6.36}$$

which then implies convergence of I and completes the proof. To this end, we introduce some further notation:

$$L_{t,T}(\underline{s}) := (bT)^{-\frac{1}{2}}\sum_{i\in H_t}K\left(\frac{i/T-u}{b}\right)\cos(\underline{s}'\underline{\widetilde{X}}_i^{(M)}(u)) \qquad \text{and} \qquad L_{t,T}^0(\underline{s}) := \zeta_t L_{t,T}(\underline{s})$$

for $\underline{s}\in[-S,S]^d$, where are $(\zeta_t)_{t\in\mathbb{N}}$ are i.i.d. Rademacher variables independent of $(\varepsilon_t)_{t\in\mathbb{Z}}$. Note that $(L_{t,T}(\underline{s}))_t$ is a sequence of independent random variables by construction. Hence, with a standard symmetrization lemma (see van der Vaart and Wellner (2000), Lemma 2.3.1) we get

$$\begin{aligned}
&E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}|\nu_T(\underline{s}_1,\underline{s}_2)|\right)\\
&= E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\sum_{t=1}^{\mu_T}[L_{t,T}(\underline{s}_1)-EL_{t,T}(\underline{s}_1)-L_{t,T}(\underline{s}_2)+EL_{t,T}(\underline{s}_2)]\right|\right)
\end{aligned}$$

$$\leq 2E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\sum_{t=1}^{\mu_T}[L^0_{t,T}(\underline{s}_1)-L^0_{t,T}(\underline{s}_2)]\right|\right).$$

Note that $\sum_{t=1}^{\mu_T} L^0_{t,T}$ has sub-Gaussian increments conditionally on $L_{1,T},\ldots,L_{\mu_T,T}$, $T\in\mathbb{N}$. This is because for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$ and $\eta>0$ we can apply Hoeffding's inequality to obtain

$$P\left(\left|\sum_{t=1}^{\mu_T} L^0_{t,T}(\underline{s}_1)-L^0_{t,T}(\underline{s}_2)\right|>\hat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2)\eta\,\middle|\,L_{1,T},\ldots,L_{\mu_T,T}\right)\leq 2\exp\left\{-\frac{\eta^2}{2}\right\}$$

with the random semimetric

$$\hat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2):=\sqrt{\sum_{t=1}^{\mu_T}\left(L_{t,T}(\underline{s}_1)-L_{t,T}(\underline{s}_2)\right)^2},\qquad \underline{s}_1,\ \underline{s}_2\in[-S,S]^d.$$

Therefore, we aim at applying a maximal inequality for sub-Gaussian processes to prove (6.36). It turns out to be easier to use a slightly modified semimetric. To this end, first note that

$$\begin{aligned}\left(L_{t,T}(\underline{s}_1)-L_{t,T}(\underline{s}_2)\right)^2&\leq\left(\left|L_{t,T}(\underline{s}_1)\right|+\left|L_{t,T}(\underline{s}_2)\right|\right)^{1-\delta}\left|L_{t,T}(\underline{s}_1)-L_{t,T}(\underline{s}_2)\right|^{1+\delta}\\&\leq 2^{1-\delta}\,|L_{t,T}|_\infty^{1-\delta}\,|L_{t,T}|_{\mathrm{Lip}}^{1+\delta}\,\rho(\underline{s}_1,\underline{s}_2)^{1+\delta}\end{aligned}\tag{6.37}$$

for $\underline{s}_1,\underline{s}_2\in[-S,S]^d$. Here, $|f|_{\mathrm{Lip}}$ denotes the Lipschitz constant of a function f. We define

$$Q_T:=2^{(1-\delta)/2}\sqrt{\sum_{t=1}^{\mu_T}|L_{t,T}|_\infty^{1-\delta}\,|L_{t,T}|_{\mathrm{Lip}}^{1+\delta}}$$

and in order to establish an upper bound for Q_T we derive

$$\|L_{t,T}\|_\infty=(bT)^{-\frac{1}{2}}\varsigma_T\|K\|_\infty\quad\text{and}\quad\|L_{t,T}\|_{\mathrm{Lip}}\leq(bT)^{-\frac{1}{2}}\|K\|_\infty\sum_{i\in H_t}\left|\underline{\widetilde{X}}_i^{(M)}(u)\right|_1,$$

which give

$$Q_T\leq C\sqrt{\frac{\varsigma_T^{1-\delta}}{bT}\sum_{t=1}^{\mu_T}\left(\sum_{i\in H_t}\left|\underline{\widetilde{X}}_i^{(M)}(u)\right|_1\right)^{1+\delta}}.$$

Using the definition of Q_T we have

$$\hat{\rho}_{T,2}(\underline{s}_1,\underline{s}_2)\leq Q_T\,\rho(\underline{s}_1,\underline{s}_2)^{(1+\delta)/2}=:\breve{\rho}_T(\underline{s}_1,\underline{s}_2).$$

Note that $\breve{\rho}_T$ is again a random semimetric since we obtain from $(1+\delta)/2\in(0,1]$

$$\breve{\rho}_T(\underline{s}_1,\underline{s}_2)\leq Q_T\left(\rho(\underline{s}_1,\underline{s}_3)+\rho(\underline{s}_3,\underline{s}_2)\right)^{(1+\delta)/2}=\breve{\rho}_T(\underline{s}_1,\underline{s}_3)+\breve{\rho}_T(\underline{s}_3,\underline{s}_2).$$

Hence, we can apply Corollary 2.2.8 of van der Vaart and Wellner (2000) as follows

$$E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\leq r_{k_T}}}\left|\sum_{t=1}^{\mu_T}[L^0_{t,T}(\underline{s}_1)-L^0_{t,T}(\underline{s}_2)]\right|\right)$$

$$
\begin{aligned}
&= E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \breve{\rho}_T(\underline{s}_1,\underline{s}_2)\le Q_T\, r_{k_T}^{(1+\delta)/2}}}\left|\sum_{t=1}^{\mu_T}[L^0_{t,T}(\underline{s}_1)-L^0_{t,T}(\underline{s}_2)]\right|\right)\\
&\le C\,E\left(\int_0^{Q_T\, r_{k_T}^{(1+\delta)/2}}\sqrt{\log D(u,[-S,S]^d,\breve{\rho}_T)}\,\mathrm{d}u\right).
\end{aligned}
$$

The packing number can be calculated as

$$
D(u,[-S,S]^d,\breve{\rho}_T)=D\left(\left(\frac{u}{Q_T}\right)^{2/(1+\delta)},[-S,S]^d,\rho\right)\le\left(\frac{2Sd}{\left(\frac{u}{Q_T}\right)^{2/(1+\delta)}}+1\right)^d.
$$

Together with Jensen's inequality, the definitions of ς_T, r_{k_T}, and Q_T, and $\log(x+1)\le x$ for $x>0$, we obtain

$$
\begin{aligned}
&E\left(\sup_{\substack{\underline{s}_1,\underline{s}_2\in[-S,S]^d\\ \rho(\underline{s}_1,\underline{s}_2)\le r_{k_T}}}\left|\sum_{t=1}^{\mu_T}[L^0_{t,T}(\underline{s}_1)-L^0_{t,T}(\underline{s}_2)]\right|\right)\\
&\le C\,E\left(\int_0^{Q_T r_{k_T}^{\frac{1+\delta}{2}}}\sqrt{\log\left(\frac{2Sd}{\left(\frac{u}{Q_T}\right)^{\frac{2}{1+\delta}}}+1\right)^d}\,\mathrm{d}u\right)\\
&\le C\,E\left(\sqrt{\frac{\varsigma_T^{1-\delta}}{bT}\sum_{t=1}^{\mu_T}\left(\sum_{i\in H_t}\left|\underline{\widetilde{X}}_i^{(M)}(u)\right|_1\right)^{1+\delta}}\right)\int_0^{r_{k_T}^{\frac{1+\delta}{2}}}\sqrt{\log\left(\frac{2Sd}{u^{\frac{2}{1+\delta}}}+1\right)^d}\,\mathrm{d}u\\
&\le C\sqrt{\frac{\varsigma_T^2}{bT}\mu_T\left(\left\|\underline{\widetilde{X}}_i^{(M)}(u)\right\|_{1+\delta}\right)^{1+\delta}}\int_0^{r_{k_T}^{\frac{1+\delta}{2}}}\sqrt{\log\left(\frac{2Sd}{u^{\frac{2}{1+\delta}}}+1\right)}\,\mathrm{d}u\\
&\le C\sqrt{\varsigma_T}\int_0^{r_{k_T}^{\frac{1+\delta}{2}}}u^{-\frac{1}{1+\delta}}\,\mathrm{d}u\\
&\le C\,(bT)^{-1/(4m)},
\end{aligned}
$$

which obviously tends to zero and thus gives (6.36) which completes the proof. ■

Proof of Theorem 3.2. By Theorem 1.5.4 and Theorem 1.5.7 of van der Vaart and Wellner (2000), we have to show convergence of the fidis and tightness in the sense of Lemma 3.3. Convergence of the fidis follows from Theorem 3.1 and tightness from Lemma 3.3. By their Addendum 1.5.8 we can deduce continuity of the sample path of the limiting process. ■

Proof of Corollary 3.1. From Assumption 2.1(ii.2) and (ii.3) we obtain

$$
\underline{\mathbf{X}}_{t,h,T}=\underline{\bar{\mu}}\left(\frac{t}{T}\right)+\sum_{j=-\infty}^{\infty}\bar{A}_{t,T}(j)\,\underline{\bar{\varepsilon}}_{t-j}+\mathcal{O}_P\left(\frac{1}{T}\right)=:\underline{\bar{X}}_{t,T}+\mathcal{O}_P\left(\frac{1}{T}\right)
$$

with $\underline{\bar{\mu}}$ and $\bar{A}_{t,T}(j)$ satisfying Assumption 2.1 again. This leads to

$$\sqrt{bT} \sup_{\underline{s}\in\mathbb{R}^{2d}} \left| \widehat{\varphi}_h(u,\underline{s}) - \frac{1}{T}\sum_{t=1}^{T} K_b\Big(\frac{t+h/2}{T} - u\Big)\, e^{i\langle \underline{s}\,,\, \underline{\bar{X}}_{t,T}\rangle} \right| = o_P(1) \quad \forall u \in (0,1).$$

Moreover, it follows from Lipschitz continuity of K that, $\forall u \in (0,1)$,

$$\sqrt{bT} \sup_{\underline{s}\in\mathbb{R}^{2d}} \left| \frac{1}{T}\sum_{t=1}^{T} \left[K_b\Big(\frac{t+h/2}{T} - u\Big) - K_b\Big(\frac{t}{T} - u\Big) \right] e^{i\langle \underline{s}\,,\, \underline{\bar{X}}_{t,T}\rangle} \right| = o_P(1).$$

Since $\underline{\bar{X}}_{t,T}$ satisfies Assumption 2.2 with $k = 1$, the assertion now follows from Theorem 3.2. ■

6.3 Proofs of Section 4

Proof of Corollary 4.1. The parameter estimators are continuous functions of $\widehat{\varphi}$. Theorem 3.1(ii) yields (pointwise) consistency of $\widehat{\varphi}$ for φ. Hence the continuous mapping theorem implies consistency of the estimators for the parameters of the stable distribution. ■

Proof of Theorem 4.1. First note that in analogy to the proof of Theorem 3.4.1 in Ushakov (1999),

$$\int \left| \varphi(u,s;\widehat{\theta}(u)) - \varphi(u,s;\theta_0(u)) \right|^2 w(s)\, ds \xrightarrow{P} 0 \quad \text{as } T \to \infty \tag{6.38}$$

follows from Theorem 3.2 and integrability of w. From this we can deduce that $\widehat{\theta}(u) \xrightarrow{P} \theta_0(u)$ as follows. First we get, that every sequence $(T_k)_k \subseteq \mathbb{N}$ contains a subsequence $(T_{kl})_l$ such that (6.38) holds a.s. for this subsequence. Assume that for a subsequence $(T_{klm})_m$ thereof, $(\widehat{\theta}_{T_{klm}}(u))_m \to \bar{\theta}(u) \neq \theta_0(u)$ *a.s.* which implies that the corresponding CFs differ on an open set, say U_0, in view of Assumption 4.1(iii). Thus, we have

$$\begin{aligned}
&\int \left| \varphi(u,s;\widehat{\theta}_{T_{klm}}(u)) - \varphi(u,s;\theta_0(u)) \right|^2 w(s) ds \\
&\geq \int_{U_0} \left| \varphi(u,s;\theta_0(u)) - \varphi(u,s;\bar{\theta}(u)) \right|^2 w(s) ds \\
&\quad - \int_{\mathbb{R}} \left| \varphi(u,s;\widehat{\theta}_{T_{klm}}(u)) - \varphi(u,s;\bar{\theta}(u)) \right|^2 w(s) ds \\
&> \eta
\end{aligned}$$

under Assumption 4.1(iii) for all large k and some $\eta > 0$, which yields a contradiction. As for assertion (ii) we abbreviate the integral on the right-hand side of (4.7) by $D_T(u;\theta(u))$. The (uniform) consistency properties of our estimator $\widehat{\varphi}(u,s)$ implies that its first derivative w.r.t. θ satisfies $\widehat{D}_T^{(1)}(u;\widehat{\theta}(u)) = 0_p$ with probability tending to one. Hence, Taylor expansion of order two gives

$$\sqrt{bT}(\widehat{\theta}(u) - \theta_0(u)) = -\sqrt{bT}\, \Big[D_T^{(2)}(u;\theta_0(u) + \xi(\widehat{\theta}(u) - \theta_0(u))) \Big]^{-1} D_T^{(1)}(u;\theta_0(u))$$

for some random $\xi \in [-1,1]$. By Assumption 4.1(iv), the order of differentiation and integration is commutable. Hence, in a neighborhood of $\theta_0(u)$ gradient and Hessian can be represented as

$$D_T^{(1)}(u;\theta(u)) = -2\int_{\mathbb{R}} \Big[(\Re\widehat{\varphi}(u,s) - \Re\varphi(u,s;\theta(u)))\Re\nabla\varphi(u,s;\theta(u)) + (\Im\widehat{\varphi}(u,s) - \Im\varphi(u,s;\theta(u)))\Im\nabla\varphi(u,s;\theta(u))\Big]\, w(s)\, ds$$

and

$$\begin{aligned} &D_T^{(2)}(u;\theta(u)) \\ &= 2\int_{\mathbb{R}} \Big(\Re\nabla\varphi(u,s;\theta(u))[\Re\nabla\varphi(u,s;\theta(u))]' + \Im\nabla\varphi(u,s;\theta(u))[\Im\nabla\varphi(u,s;\theta(u))]' \\ &\quad - (\Re\widehat{\varphi}(u,s) - \Re\varphi(u,s;\theta(u)))\Re H_\varphi(u,s;\theta(u)) \\ &\quad - (\Im\widehat{\varphi}(u,s) - \Im\varphi(u,s;\theta(u)))\Im H_\varphi(u,s;\theta(u))\Big)\, w(s)\, ds, \end{aligned}$$

respectively. By the first part and Theorem 3.2 $D_T^{(2)}(u;\theta_0(u) + \xi(\widehat{\theta}(u) - \theta_0(u))) \xrightarrow{P} 2D_0$, which is invertible by assumption. Hence, it remains to show that

$$\sqrt{bT}\, D_T^{(1)}(u;\theta_0(u)) \xrightarrow{d} 2D_0\, Z_{MDE}.$$

Applying Theorem 3.2 and the continuous mapping theorem, we obtain

$$\begin{aligned} R_{S,T} :=& \sqrt{bT}\int_{[-S,S]^d} \Big[(\Re\widehat{\varphi}(u,s) - \Re\varphi(u,s;\theta_0(u)))\Re\nabla\varphi(u,s;\theta_0(u)) \\ &+ (\Im\widehat{\varphi}(u,s) - \Im\varphi(u,s;\theta_0(u)))\Im\nabla\varphi(u,s;\theta_0(u))\Big]\, w(s)\, ds \\ \xrightarrow{d} R_S :=& \int_{[-S,S]^d} \left[Z_1(u,s)\,\Re\nabla\varphi(u,s;\theta_0(u)) + Z_2(u,s)\,\Im\nabla\varphi(u,s;\theta_0(u))\right]\, w(s)\, ds \end{aligned}$$

for any $S \in (0,\infty)$, where $Z = (Z_1, Z_2)'$ is the limiting process in Theorem 3.2 applied to $\varphi(u,s) = \varphi(u,s;\theta_0)$. It follows from a usual Riemann approximation of the integral that the limit R_S is a centered normal random variable. By (4.10), $\mathrm{Var}(R_S) \to \mathrm{Var}(D_0\, Z_{MDE}) < \infty$ as $S \to \infty$, which implies that

$$R_S \xrightarrow{d} D_0\, Z_{MDE}.$$

Now, the assertion follows from Proposition 6.3.9 in Brockwell and Davis (1991) if additionally

$$\lim_{S\to\infty} \limsup_{T\to\infty} P\left(|R_{S,T} - \sqrt{bT}\, D_T^{(1)}(u;\theta_0(u))\Big)\,| > \eta) = 0.$$

In view of (4.10), Lemma 3.1(i) implies

$$\lim_{S\to\infty} \limsup_{T\to\infty} E\left|R_{S,T} - \sqrt{bT}\, D_T^{(1)}(u;\theta_0(u))\right|^2 = 0,$$

which then completes the proof. ■

Proof of Lemma 4.2. By expanding the squared term of the integrand in (4.17), we get

$$|\kappa_T\widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)|^2$$

$$
\begin{aligned}
&= |\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)|^2 - \kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)\overline{\widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)} \\
&\quad -\kappa_T \overline{\widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)}\widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2) + |\widehat{\varphi}_{Y;0}(u,\underline{s}_1)\widehat{\varphi}_{Z;h}(u,\underline{s}_2)|^2 \\
&= I + II + III + IV.
\end{aligned}
$$

From (4.18) and expanding $\exp(i\cdot) = \cos(\cdot) + i\sin(\cdot)$, we get for the first term

$$
I = \frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T-h}\prod_{j=1}^{4} K_b\Big(\frac{t_j+h/2}{T}-u\Big)\cos\big(\underline{s}_1'(\underline{Y}_{t_3,T}-\underline{Y}_{t_4,T})\big)\cos\big(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T})\big) + R_I,
$$

where R_I is a remainder term that leads to vanishing terms for the integral in (4.17) (note that $\sin(\cdot)$ is an odd function). By using the identity

$$
\cos(u)\cos(v) = 1-(1-\cos(u))-(1-\cos(v))+(1-\cos(u))(1-\cos(v))
$$

we get

$$
\begin{aligned}
I &= R_I + \frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T-h}\prod_{j=1}^{4} K_b\Big(\frac{t_j+h/2}{T}-u\Big) \\
&\quad \times\Big\{1-\big(1-\cos\big(\underline{s}_1'(\underline{Y}_{t_3,T}-\underline{Y}_{t_4,T})\big)\big)-\big(1-\cos\big(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T})\big)\big) \\
&\quad + \big(1-\cos\big(\underline{s}_1'(\underline{Y}_{t_3,T}-\underline{Y}_{t_4,T})\big)\big)\big(1-\cos\big(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T})\big)\big)\Big\} \\
&=: I_1 - I_2 - I_3 + I_4 + R_I.
\end{aligned}
$$

The same calculation for terms II, III and IV leads to similar expressions, where the first three terms cancel out, i.e. $I_j + II_j + III_j + IV_j = 0$ for $j = 1,2,3$ and $II_4 = III_4$. Note that the cancellation is due to the inclusion of the factor κ_T in the definition of (4.16). Altogether, this leads to

$$
\begin{aligned}
&|\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)|^2 \\
=&\frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T-h}\prod_{j=1}^{4} K_b\Big(\frac{t_j+h/2}{T}-u\Big) \\
&\times\Big\{\big(1-\cos(\underline{s}_1'(\underline{Y}_{t_3,T}-\underline{Y}_{t_4,T}))\big)\big(1-\cos(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}))\big) \\
&-2\big(1-\cos(\underline{s}_1'(\underline{Y}_{t_2,T}-\underline{Y}_{t_4,T}))\big)\big(1-\cos(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}))\big) \\
&+\big(1-\cos(\underline{s}_1'(\underline{Y}_{t_1,T}-\underline{Y}_{t_2,T}))\big)\big(1-\cos(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}))\big)\Big\} + R,
\end{aligned}
\tag{6.39}
$$

where R is a remainder term that leads to vanishing terms for the integral in (4.17). Now, we plug the above into (4.17) and make use of Lemma 4.1 to get the claimed result. ■

Proof of Theorem 4.2. To prove consistency of $\widehat{\mathcal{V}}^2_{Y,Z}(u;h)$ we proceed in three steps and show:

1. For $\eta \in (0,1)$

$$
\begin{aligned}
\widehat{\mathcal{V}}^2_{Y,Z;\eta}(u;h) &:= \int_{D_\eta} |\kappa_T\, \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)|^2\, d\omega \\
\xrightarrow{P} \mathcal{V}^2_{Y,Z;\eta}(u;h) &:= \int_{D_\eta} |\varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;h}(u;\underline{s}_2)|^2\, d\omega,
\end{aligned}
$$

where $D_\eta = \{(\underline{s}_1',\underline{s}_2')' \in \mathbb{R}^{p+q} \mid \eta \le |\underline{s}_1|_2 \le 1/\eta,\ \eta \le |\underline{s}_2|_2 \le 1/\eta\}$.

2. $\mathcal{V}^2_{Y,Z;\eta}(u;h) \longrightarrow \mathcal{V}^2_{Y,Z}(u;h)$ with $\eta \to 0$.

3. $\lim_{\eta\to 0} \limsup_{T\to\infty} P\left(\left|\widehat{\mathcal{V}}^2_{Y,Z;\eta}(u;h) - \widehat{\mathcal{V}}^2_{Y,Z}(u;h)\right| > \epsilon\right) = 0 \quad \forall\, \epsilon > 0.$

Then, the assertion for distance covariance can be deduced from Proposition 6.3.9 in Brockwell and Davis (1991) and it remains to consider the distance correlation. Under the assumptions of the theorem $\mathcal{V}^2_Y(u;0)\,\mathcal{V}^2_Z(u;0) > \bar{\eta}$ for some $\bar{\eta} > 0$. Hence, with probability tending to one, we have $\mathbb{1}(\widehat{\mathcal{V}}^2_Y(u;0)\,\widehat{\mathcal{V}}^2_Z(u;0) > 0) = 1$. Thus, stochastic convergence of $\widehat{\mathcal{R}}^2_{Y,Z}(u;h)$ follows immediately from convergence of the distance covariance and it remains to carry out steps 1 to 3 mentioned above.
Step 1. We rewrite

$$\begin{aligned}\widehat{\mathcal{V}}^2_{Y,Z;\eta}(u;h) = \int_{D_\eta} \big|&\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)\\ &+ \varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;h}(u;\underline{s}_2)\\ &+ \varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;h}(u;\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)\big|^2\, d\omega.\end{aligned}$$

By Cauchy-Schwarz inequality and recognizing that

$$\int_{D_\eta} \left|\varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;h}(u;\underline{s}_2)\right|^2\, d\omega < \infty,$$

it remains to show that

$$\int_{D_\eta} \left|\kappa_T\, \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y,Z,h}(u;\underline{s}_1,\underline{s}_2)\right|^2 d\omega \xrightarrow{P} 0 \tag{6.40}$$

and

$$\int_{D_\eta} \left|\varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;h}(u;\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)\right|^2 d\omega \xrightarrow{P} 0. \tag{6.41}$$

For (6.40) note that

$$\begin{aligned}&\int_{D_\eta} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)\right|^2 d\omega\\ &\le 2\int_{D_\eta} (\kappa_T - 1)^2 |\widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)|^2\, d\omega + 2\int_{D_\eta} \left|\widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2)\right|^2 d\omega.\end{aligned} \tag{6.42}$$

The first summand on the right-hand side tends to zero in probability since $\kappa_T \to 1$ and $\widehat{\varphi}_{Y,Z;h}$ is bounded (uniformly in T). Following the lines of the proof of Corollary 4.1, we obtain that $\sqrt{bT}(\widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \varphi_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2))_{(\underline{s}_1',\underline{s}_2')'\in D_\eta}$ converges in distribution to a complex centered Gaussian process with continuous sample paths. Together with the continuous mapping theorem this gives asymptotic negligibility of the second summand in (6.42). With the same arguments (6.41) can be obtained.
Step 2. The assertion is an immediate consequence of the monotone convergence theorem.

Step 3. We follow the lines of the proof of Theorem 2 in Szekely, Rizzo and Bakirov (2007). First, note that

$$\widehat{\mathcal{V}}^2_{Y,Z}(u;h) - \widehat{\mathcal{V}}^2_{Y,Z;\eta}(u;h) = \int_{\mathbb{R}^{p+q}\setminus D_\eta} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)\right|^2 d\omega.$$

and

$$\begin{aligned}\mathbb{R}^{p+q}\setminus D_\eta =& \{(\underline{s}_1',\underline{s}_2')' \in \mathbb{R}^{p+q} \mid |\underline{s}_1|_2 < \eta\} \cup \{(\underline{s}_1',\underline{s}_2')' \in \mathbb{R}^{p+q} \mid |\underline{s}_1|_2 > 1/\eta\} \\ &\cup \{(\underline{s}_1',\underline{s}_2')' \in \mathbb{R}^{p+q} \mid |\underline{s}_2|_2 < \eta\} \cup \{(\underline{s}_1',\underline{s}_2')' \in \mathbb{R}^{p+q} \mid |\underline{s}_2|_2 > 1/\eta\}.\end{aligned}$$

Hence, for symmetry reasons it suffices to show that $\forall \varepsilon > 0$

$$\lim_{\eta\to 0}\limsup_{T\to\infty} P\left(\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||s_1|_2>1/\eta\}} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)\right|^2 d\omega > \varepsilon\right) \quad (6.43)$$

and

$$\lim_{\eta\to 0}\limsup_{T\to\infty} P\left(\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||s_1|_2<\eta\}} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u;\underline{s}_1,\underline{s}_2) - \widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;h}(u;\underline{s}_2)\right|^2 d\omega > \varepsilon\right) \quad (6.44)$$

are equal to zero. For (6.43) we invoke expansion (6.39) of the integrand and Lemma 4.1

$$\begin{aligned}
&\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2>1/\eta\}} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u,(\underline{s}_1,\underline{s}_2)) - \widehat{\varphi}_{Y;0}(u,\underline{s}_1)\widehat{\varphi}_{Z;h}(u,\underline{s}_2)\right|^2 d\omega \\
=\ & \int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2>1/\eta\}} \Big|\frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T-h}\prod_{j=1}^{4} K_b\Big(\frac{t_j+h/2}{T}-u\Big) \\
&\times\Big\{ \big(1-\cos(\underline{s}_1'(\underline{Y}_{t_3,T}-\underline{Y}_{t_4,T}))\big)\big(1-\cos(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}))\big) \\
&-2\big(1-\cos(\underline{s}_1'(\underline{Y}_{t_2,T}-\underline{Y}_{t_4,T}))\big)\big(1-\cos\big(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T})\big)\big) \\
&+\big(1-\cos(\underline{s}_1'(\underline{Y}_{t_1,T}-\underline{Y}_{t_2,T}))\big)\big(1-\cos(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}))\big)\Big\}\Big| d\omega \\
\le\ & 8\int_{|\underline{s}_1|_2>1/\eta}\frac{1}{c_p\,|\underline{s}_1|_2^{1+p}}d\underline{s}_1 \\
&\times\int_{\mathbb{R}^q}\frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T-h}\prod_{j=1}^{4}K_b\Big(\frac{t_j+h/2}{T}-u\Big)\big(1-\cos\big(\underline{s}_2'(\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T})\big)\big)\frac{1}{c_q\,|\underline{s}_2|_2^{1+q}}d\underline{s}_2 \\
\le\ & C\int_{|\underline{s}_1|_2>1/\eta}\frac{1}{|\underline{s}_1|_2^{1+p}}d\underline{s}_1\,\frac{1}{T^2}\sum_{t_3,t_4=1}^{T-h}\prod_{j=3}^{4}K_b\Big(\frac{t_j+h/2}{T}-u\Big)|\underline{Z}_{t_3+h,T}-\underline{Z}_{t_4+h,T}| \\
\le\ & C\int_{|\underline{s}_1|_2>1/\eta}\frac{1}{|\underline{s}_1|_2^{1+p}}d\underline{s}_1\sum_{t=1}^{T-h}K_b\Big(\frac{t+h/2}{T}-u\Big)|\underline{Z}_{t+h,T}|.
\end{aligned}$$

Now, Markov's inequality gives

$$(6.43)\ \le\ C\lim_{\eta\to 0}\int_{|\underline{s}_1|_2>1/\eta}\frac{1}{|\underline{s}_1|_2^{1+p}}d\underline{s}_1\ \limsup_{T\to\infty}\sum_{t=1}^{T-h}K_b\Big(\frac{t+h/2}{T}-u\Big)E|\underline{Z}_{t+h,T}|\ =\ 0.$$

The verification of (6.44) is more involved. We denote by $\underline{\bar{X}}(u) = (\underline{\bar{Y}}'(u), \underline{\bar{Z}}'(u))'$ a copy of $\underline{\widetilde{X}}_1(u)$ which is independent of the DGP $(\underline{X}_{t,T})_{t=1}^T,\ T \in \mathbb{N}$. Using the inequalities of Cauchy-Schwarz and Jensen, we obtain

$$
\begin{aligned}
&\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2<\eta\}} \left|\kappa_T \widehat{\varphi}_{Y,Z;h}(u,(\underline{s}_1,\underline{s}_2)) - \widehat{\varphi}_{Y;0}(u,\underline{s}_1)\widehat{\varphi}_{Z;h}(u,\underline{s}_2)\right|^2 d\omega \\
&= \int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2<\eta\}} \Bigg| \frac{\kappa_T}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)(e^{i\underline{s}_1'\underline{Y}_{t,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)})(e^{i\underline{s}_2'\underline{Z}_{t+h,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)}) \\
&\quad - \frac{1}{T^2}\sum_{t_1,t_2=1}^{T-h} K_b\Big(\frac{t_1+h/2}{T}-u\Big)K_b\Big(\frac{t_2+h/2}{T}-u\Big)(e^{i\underline{s}_1'\underline{Y}_{t_1,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)})(e^{i\underline{s}_2'\underline{Z}_{t_2+h,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)})\Bigg|^2 d\omega \\
&\le 2\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2<\eta\}} \left(\frac{\kappa_T}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_1'\underline{Y}_{t,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)}\right|^2\right) \\
&\quad \times \left(\frac{\kappa_T}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_2'\underline{Z}_{t+h,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)}\right|^2\right) d\omega \\
&\quad + 2\int_{\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}||\underline{s}_1|_2<\eta\}} \frac{1}{T^4}\left|\sum_{t_1=1}^{T-h} K_b\Big(\frac{t_1+h/2}{T}-u\Big)(e^{i\underline{s}_1'\underline{Y}_{t_1,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)})\right|^2 \\
&\quad \times \left|\sum_{t_2=1}^{T-h} K_b\Big(\frac{t_2+h/2}{T}-u\Big)(e^{i\underline{s}_2'\underline{Z}_{t_2+h,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)})\right|^2 d\omega \\
&\le 4\kappa_T^2 \int_{|\underline{s}_1|_2\le\eta} \frac{1}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_1'\underline{Y}_{t,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)}\right|^2 \frac{1}{c_p|\underline{s}_1|_2^{1+p}} d\underline{s}_1 \\
&\quad \times \int_{\underline{s}_2\in\mathbb{R}^q} \frac{1}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_2'\underline{Z}_{t+h,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)}\right|^2 \frac{1}{c_q|\underline{s}_2|_2^{1+q}} d\underline{s}_2.
\end{aligned}
$$

With the same arguments as in Szekely, Rizzo and Bakirov (2007, page 2777f.), we obtain

$$
\begin{aligned}
&\int_{\underline{s}_2\in\mathbb{R}^q} \frac{1}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_2'\underline{Z}_{t,T}} - Ee^{i\underline{s}_2'\underline{\bar{Z}}(u)}\right|^2 \frac{1}{c_q|\underline{s}_2|_2^{1+q}} d\underline{s}_2 \\
&\le \frac{2}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left(|\underline{Z}_{t+h,T}| + E|\underline{\bar{Z}}(u)|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{|\underline{s}_1|_2\le\eta} \frac{1}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)\left|e^{i\underline{s}_1'\underline{Y}_{t,T}} - Ee^{i\underline{s}_1'\underline{\bar{Y}}(u)}\right|^2 \frac{1}{c_p|\underline{s}_1|_2^{1+p}} d\underline{s}_1 \\
&\le \frac{2}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big) E\big(|\underline{Y}_{t,T} - \underline{\bar{Y}}(u)| G(|\underline{Y}_{t,T} - \underline{\bar{Y}}(u)|\eta)\,|\,\underline{Y}_{t,T}\big),
\end{aligned}
$$

where

$$
G(y) = \int_{|\underline{z}|_2\le y} \frac{1-\cos(z_1)}{|\underline{z}|_2^{1+p}}\, d\underline{z}, \quad \underline{z} = (z_1,\ldots,z_p)'.
$$

Summing up, we have for (6.44) by Markov's inequality and Lebesgue's dominated convergence theorem

(6.44)

$$
\begin{aligned}
&\leq \lim_{\eta\to 0}\limsup_{T\to\infty} P\Bigg(\frac{16}{T^2}\,\kappa_T^2 \sum_{t_1=1}^{T-h} K_b\Big(\frac{t_1+h/2}{T}-u\Big)\,\big(|\underline{Z}_{t_1+h,T}|+E|\bar{\underline{Z}}(u)|\big)\\
&\qquad\qquad \times \sum_{t_2=1}^{T-h} K_b\Big(\frac{t_2+h/2}{T}-u\Big)\,E\big(|\underline{Y}_{t_2,T}-\bar{\underline{Y}}(u)|G(|\underline{Y}_{t_2,T}-\bar{\underline{Y}}(u)|\eta)\,|\,\underline{Y}_{t_2,T}\big)>\varepsilon\Bigg)\\
&\leq \eta_M + \lim_{\eta\to 0}\limsup_{T\to\infty} P\Bigg(\frac{M}{T}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)E\big(|\underline{Y}_{t,T}-\bar{\underline{Y}}(u)|G(|\underline{Y}_{t,T}-\bar{\underline{Y}}(u)|\eta)\,|\,\underline{Y}_{t,T}\big)>\varepsilon\Bigg)\\
&\leq \eta_M + \lim_{\eta\to 0}\limsup_{T\to\infty} \frac{M}{T\,\varepsilon}\sum_{t=1}^{T-h} K_b\Big(\frac{t+h/2}{T}-u\Big)E\big(|\underline{Y}_{t,T}-\bar{\underline{Y}}(u)|G(|\underline{Y}_{t,T}-\bar{\underline{Y}}(u)|\eta)\big)\\
&\leq \eta_M
\end{aligned}
$$

with $\eta_M \to 0$ as $M\to\infty$. This leads to the desired result as G is bounded and $G(y)\to 0$ with $y\to 0$. ■

Proof of Theorem 4.3. We follow the lines of the proof of Theorem 3.2(2) in Davis, Matsui, Mikosch, and Wan (2016) and define

$$
\begin{aligned}
\widehat{C}_T(\underline{s}_1,\underline{s}_2;u) &= \kappa_T\widehat{\varphi}_{Y,Z;0}(u;\underline{s}_1,\underline{s}_2)-\widehat{\varphi}_{Y;0}(u;\underline{s}_1)\widehat{\varphi}_{Z;0}(u;\underline{s}_2),\\
C(\underline{s}_1,\underline{s}_2;u) &= \varphi_{Y,Z;0}(u;\underline{s}_1,\underline{s}_2)-\varphi_{Y;0}(u;\underline{s}_1)\varphi_{Z;0}(u;\underline{s}_2).
\end{aligned}
$$

To prove that

$$
\begin{aligned}
\sqrt{bT}(\widehat{\mathcal{V}}^2_{Y,Z}(u;0)-\mathcal{V}^2_{Y,Z}(u;0)) &= \sqrt{bT}\int_{\mathbb{R}^p\times\mathbb{R}^q} |\widehat{C}_T(\underline{s}_1,\underline{s}_2;u)|^2-|C(\underline{s}_1,\underline{s}_2;u)|^2 d\omega\\
&\xrightarrow{d} \int_{\mathbb{R}^p\times\mathbb{R}^q} G(\underline{s}_1,\underline{s}_2;u)d\omega,
\end{aligned}
$$

note that by Assumption 2.2

$$
\begin{aligned}
\sqrt{bT}(\widehat{\mathcal{V}}^2_{Y,Z}(u;0)-\mathcal{V}^2_{Y,Z}(u;0)) =& \sqrt{bT}\int_{\mathbb{R}^p\times\mathbb{R}^q} |\widehat{C}_T(\underline{s}_1,\underline{s}_2;u)|^2-|C(\underline{s}_1,\underline{s}_2;u)|^2 d\omega\\
=& \sqrt{bT}\int_{\mathbb{R}^p\times\mathbb{R}^q} |\widehat{C}_T(\underline{s}_1,\underline{s}_2;u)|^2-|\kappa_T^2 C(\underline{s}_1,\underline{s}_2;u)|^2 d\omega + O((bT)^{-1/2}).
\end{aligned}
$$

Now, we apply Proposition 6.3.9 in Brockwell and Davis (1991) and proceed in three steps. We show:

1. For $\eta\in(0,1)$ and $D_\eta=\{(\underline{s}_1',\underline{s}_2')'\in\mathbb{R}^{p+q}\mid \eta\leq|\underline{s}_1|_2\leq 1/\eta,\ \eta\leq|\underline{s}_2|_2\leq 1/\eta\}$ we have

$$
\sqrt{bT}\int_{D_\eta} |\widehat{C}_T(\underline{s}_1,\underline{s}_2;u)|^2-|\kappa_T^2 C(\underline{s}_1,\underline{s}_2;u)|^2 d\omega \xrightarrow{d} \int_{D_\eta} G(\underline{s}_1,\underline{s}_2;u)d\omega.
$$

2. For $\eta \to 0$
$$\mathcal{G}_\eta := \int_{D_\eta} G(\underline{s}_1, \underline{s}_2; u) d\omega \xrightarrow{d} \int_{\mathbb{R}^p \times \mathbb{R}^q} G(\underline{s}_1, \underline{s}_2; u) d\omega =: \mathcal{G}.$$

3. $\lim_{\eta \downarrow 0} \limsup_{T \to \infty} P\left(\sqrt{bT} \left| \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} |\widehat{C}_T(\underline{s}_1, \underline{s}_2; u)|^2 - |\kappa_T^2 C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \right| > \epsilon\right) = 0 \quad \forall \epsilon > 0.$

Step 1. The assertion follows from Theorem 3.2 with the same arguments as in the proof of Theorem 3.2(2) in Davis, Matsui, Mikosch, and Wan (2016). To this end, note that
$$\begin{aligned}
&\sqrt{bT} \int_{D_\eta} |\widehat{C}_T(\underline{s}_1, \underline{s}_2; u)|^2 - |\kappa_T^2\, C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \\
=&\sqrt{bT} \int_{D_\eta} [\widehat{C}_T(\underline{s}_1, \underline{s}_2; u) - \kappa_T^2 C(\underline{s}_1, \underline{s}_2; u)]\, \widehat{C}_T(-\underline{s}_1, -\underline{s}_2; u) \\
&+ [\widehat{C}_T(-\underline{s}_1, -\underline{s}_2; u) - \kappa_T^2 C(-\underline{s}_1, -\underline{s}_2; u)]\, \kappa_T^2 C(\underline{s}_1, \underline{s}_2; u) d\omega.
\end{aligned}$$
and $|\kappa_T - 1| = O((bT)^{-1})$.

Step 2: First, note that Steps 1. and 3. assure existence of the limit distribution of $\sqrt{bT}(\widehat{\mathcal{V}}^2_{Y,Z}(u;0) - \mathcal{V}^2_{Y,Z}(u;0))$; see Theorem 2 in Dehling, Durieu, and Volny (2009). Since we deal with centered Gaussian objects G_η, it suffices to show that $E\mathcal{G}_\eta^2$ converge as $\eta \to 0$. Obviously, boundedness of the covariance function of $\bar{Z}$ and of $\mathcal{V}^2_{X,Y}(u;0)$ imply that
$$E\mathcal{G}_\eta^2 \le 4 \int_{D_\eta} E|\bar{Z}(\underline{s}_1, \underline{s}_2; u)|^2\, |C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \le 4 \sup_{\underline{s}_1, \underline{s}_2} E|\bar{Z}(\underline{s}_1, \underline{s}_2; u)|^2\, \mathcal{V}^2_{X,Y}(u;0) < \infty,$$
that is, all variances are uniformly bounded. Moreover, for $\widetilde{\eta} < \eta$
$$E(\mathcal{G}_{\widetilde{\eta}} - \mathcal{G}_\eta)^2 \le 4 \int_{\mathbb{R}^p \times \mathbb{R}^q \setminus D_\eta} E|\bar{Z}(\underline{s}_1, \underline{s}_1; u)|^2\, |C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \longrightarrow 0$$
as $\eta \to 0$. Hence, $(\mathcal{G}_{\eta_n})_n$ forms a Cauchy sequence in L_2 for any null sequence $(\eta_n)_n$ which finishes the proof of Step 2.

Step 3: For symmetry reasons we only consider
$$\sqrt{bT} \int_{\bar{D}^0_\eta} |\widehat{C}_T(\underline{s}_1, \underline{s}_2; u)|^2 - |\kappa_T^2 C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega$$
with $\bar{D}^0_\eta = \{(\underline{s}_1', \underline{s}_2')' \in \mathbb{R}^{p+q} \mid |\underline{s}_1|_2 \notin (\eta, 1/\eta)\}$. Let $(\widetilde{\underline{X}}_t^{(j)}(u))_t$, $j = 1, \ldots, 4$, denote independent copies of $(\widetilde{\underline{X}}_t(u))_t$ being independent of $(\underline{X}_{t,T})_t$, too. Similarly to Step 2 of the previous proof, we obtain
$$\begin{aligned}
&\sqrt{bT} \int_{\bar{D}^0_\eta} |\widehat{C}_T(\underline{s}_1, \underline{s}_2; u)|^2 - |\kappa_T^2 C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \\
=&\sqrt{bT} \int_{\bar{D}^0_\eta} \frac{1}{T^4} \sum_{t_1,t_2,t_3,t_4=1}^{T} \prod_{j=1}^{4} K_b\Big(\frac{t_j}{T} - u\Big) \Big(\big(1 - \cos(\underline{s}_2'(\underline{Z}_{t_3,T} - \underline{Z}_{t_4,T}))\big) \\
&\times \Big\{ \big(1 - \cos(\underline{s}_1'(\underline{Y}_{t_3,T} - \underline{Y}_{t_4,T}))\big) - 2\big(1 - \cos(\underline{s}_1'(\underline{Y}_{t_2,T} - \underline{Y}_{t_4,T}))\big) + \big(1 - \cos(\underline{s}_1'(\underline{Y}_{t_1,T} - \underline{Y}_{t_2,T}))\big) \Big\} \\
&- E\Big[\Big(1 - \cos(\underline{s}_2'(\widetilde{\underline{Z}}_0^{(3)}(u) - \widetilde{\underline{Z}}_0^{(4)}(u)))\Big) \times \Big\{ \Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(3)}(u) - \widetilde{\underline{Y}}_0^{(4)}(u)))\Big) \\
&- 2\Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(2)}(u) - \widetilde{\underline{Y}}_0^{(4)}(u)))\Big) + \Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(1)}(u) - \widetilde{\underline{Y}}_0^{(2)}(u)))\Big) \Big\} \Big] \Big) d\omega.
\end{aligned}$$

Similarly to Lemma 2.1, we obtain from $\|\varepsilon_0\|_1^4 < \infty$

$$(E(|\underline{Z}_{t_3,T} - \underline{Z}_{t_4,T}|_1 - |\widetilde{\underline{Z}}_{t_3}(u) - \widetilde{\underline{Z}}_{t_4}(u)|_1)^2)^{1/2} = O(T^{-1} + b).$$

and

$$\begin{aligned}
&\left[E\left(\int_{\underline{s}_1 \notin (\eta, 1/\eta)} \Big(\cos(\underline{s}_1'(\underline{Y}_{t_3,T} - \underline{Y}_{t_4,T})) - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_{t_3}(u) - \widetilde{\underline{Y}}_{t_4}(u)))\Big)\right)^2\right]^{1/2} \\
\leq &\left[E\left(\int_{\underline{s}_1 < \eta} \frac{|\underline{Y}_{t_3,T} - \underline{Y}_{t_4,T} - \widetilde{\underline{Y}}_{t_3}(u) + \widetilde{\underline{Y}}_{t_4}(u)|_1^{1+\delta_1/2}}{|\underline{s}_1|_2^{p-\delta_1/2}} d\underline{s}_1\right)^2\right]^{1/2} \\
&+ \left[E\left(\int_{\underline{s}_1 > 1/\eta} \frac{|\underline{Y}_{t_3,T} - \underline{Y}_{t_4,T} - \widetilde{\underline{Y}}_{t_3}(u) + \widetilde{\underline{Y}}_{t_4}(u)|_1^{1-\delta_1/2}}{|\underline{s}_1|_2^{p+\delta_1/2}} d\underline{s}_1\right)^2\right]^{1/2} \\
= &O((T^{-1} + b)^{1-\delta_1/2}).
\end{aligned} \tag{6.45}$$

This gives $\sqrt{bT}O((T^{-1} + b)^{1-\delta_1/2}) = o(1)$. Hence, we get from Lemma 4.1

$$\begin{aligned}
&\sqrt{bT}\int_{\bar{D}_\eta^0} |\widehat{C}_T(\underline{s}_1, \underline{s}_2; u)|^2 - |\kappa_T^2 C(\underline{s}_1, \underline{s}_2; u)|^2 d\omega \\
\leq &o(1) + \sqrt{bT}\frac{1}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\prod_{j=1}^{4} K_b\Big(\frac{t_j}{T} - u\Big)\Big(|\widetilde{\underline{Z}}_{t_3}(u) - \widetilde{\underline{Z}}_{t_4}(u)|_2 \\
&\times \int_{\underline{s}_1 \notin (\eta, 1/\eta)} \frac{c_p}{|\underline{s}_1|_2^{1+p}}\Big\{\Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_{t_3}(u) - \widetilde{\underline{Y}}_{t_4}(u)))\Big) - 2\Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_{t_2}(u) - \widetilde{\underline{Y}}_{t_4}(u)))\Big) \\
&+ \Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_{t_1}(u) - \widetilde{\underline{Y}}_{t_2}(u)))\Big)\Big\} d\underline{s}_1 \\
&- E\Big[|\widetilde{\underline{Z}}_0^{(3)}(u) - \widetilde{\underline{Z}}_0^{(4)}(u)|_2 \times \int_{\underline{s}_1 \notin (\eta, 1/\eta)} \frac{c_p}{|\underline{s}_1|_2^{1+p}}\Big\{\Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(3)}(u) - \widetilde{\underline{Y}}_0^{(4)}(u)))\Big) \\
&- 2\Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(2)}(u) - \widetilde{\underline{Y}}_0^{(4)}(u)))\Big) + \Big(1 - \cos(\underline{s}_1'(\widetilde{\underline{Y}}_0^{(1)}(u) - \widetilde{\underline{Y}}_0^{(2)}(u)))\Big)\Big\} d\underline{s}_1\Big]\Big) \\
=: &o_P(1) + \frac{\sqrt{bT}}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\prod_{j=1}^{4} K_b\Big(\frac{t_j}{T} - u\Big) R_{\eta;t_1,\ldots,t_4}.
\end{aligned}$$

with an obvious definition of $R_{\eta;t_1,\ldots,t_4}$. We will show that

$$\lim_{\eta \downarrow 0}\limsup_{T\to\infty} E\left(\frac{\sqrt{bT}}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\prod_{j=1}^{4} K_b\Big(\frac{t_j}{T} - u\Big) R_{\eta;t_1,\ldots,t_4}\right)^2 = 0$$

Note that $ER_{\eta;t_1,\ldots,t_4} = 0$ for distinct values $t_1, \ldots, t_4$. Finally, we obtain

$$\lim_{\eta\downarrow 0}\limsup_{T\to\infty} E\left(\frac{\sqrt{bT}}{T^4}\sum_{t_1,t_2,t_3,t_4=1}^{T}\prod_{j=1}^{4} K_b\Big(\frac{t_j}{T}-u\Big)\, R_{\eta;t_1,\ldots,t_4}\right)^2$$

$$=\lim_{\eta\downarrow 0}\limsup_{T\to\infty}\frac{bT}{T^8}\sum_{\substack{t_1,\ldots,t_8=1\\ |\{t_1,\ldots,t_8\}|=7}}^{T}\prod_{j=1}^{8} K_b\Big(\frac{t_j}{T}-u\Big)\, ER_{\eta;t_1,\ldots,t_4}R_{\eta;t_7,\ldots,t_8}$$

$$\leq \lim_{\eta\downarrow 0} C\,\left(E|\underline{\widetilde{Z}}_0^{(3)}(u)-\underline{\widetilde{Z}}_0^{(4)}(u)|_2^4\right)^{1/4}\left(E\left\{\int_{\underline{s}_1\notin(\eta,1/\eta)}\frac{c_p}{|\underline{s}_1|_2^{1+p}}\Big(1-\cos(\underline{s}_1'(\underline{\widetilde{Y}}_0^{(1)}(u)-\underline{\widetilde{Y}}_0^{(2)}(u)))\Big)\,d\underline{s}_1\right\}^4\right)^{1/4}.$$

which vanishes with the similar approximations of the remaining integral as in (6.45) under the assumption that $E|\varepsilon_0|_1^{4+\delta_2} < \infty$. ■

References

Arcones, M.A. and Yu, B. (1994). Central limit theorems for empirical and U-processes of stationary mixing sequences. *Journal of Theoretical Probability* **7**, 47–71.

Brockwell, P. J. and Davis, R. A. (1991). *Time Series: Theory and Methods.* Springer, New York.

Dehling, H., Durieu, O., and Volny, D. (2009). New techniques for empirical processes of dependent data. *Stochastic Processes and their Applications* **119**, 3699–3718.

Romano, J. P. and Wolf, M. (2000). A likelihood approximation for locally stationary processes. *Statist. Probab. Lett* **47**, 115–124.

Van der Vaart, A. W. and Wellner, J. A. (2000). *Weak Convergence and Empirical Processes. With Applications to Statistics.* New York: Springer.