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A fluctuation test for constant Spearman's rho

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by

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Testing for constant Spearman's rho

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Abstract

We propose a CUSUM type test for constant correlation that goes beyond a previously suggested correlation constancy test by considering Spearman's rho in arbitrary dimensions. By using copula-based expressions, we simultaneously extend a previously suggested copula constancy test. We calculate the asymptotic null distribution using an invariance principle for the sequential empirical copula process. The limit distribution is free of nuisance parameters and critical values can be obtained without bootstrap techniques. We give a local power result and analyze the test's behavior in small samples.

Keywords: Copula, Mixing, Multivariate sequential empirical process, Robustness, Structural break

1. Introduction

Recently, Wied, Krämer and Dehling (2010) proposed a fluctuation test for constant correlation based on the Bravais-Pearson correlation coefficient. The test, which will be referred to as BPC test in the following, is useful e.g. in financial econometrics when a practitioner wants to analyze if correlations of asset returns change in time, see e.g. Longin and Solnik (1995) and Krishan et al. (2009) for the relevance of this question. It complements former approaches by e.g. Galeano and Peña (2007) and Aue et al. (2009).

This paper presents a fluctuation test for constant correlation based on Spearman's rho and the sample version of it. In many situations, e.g. if the data is non-elliptical, the Bravais-Pearson correlation may not be an appropriate measure for dependence. It is e.g. confined to measuring linear dependence, while the rank-based dependence measure Spearman's rho quantifies monotone dependence. Spearman's rho is probably the most common rank-based dependence measure in economic and social sciences, see e.g. Gaißler and Schmid (2010), who propose tests for equality of rank correlations, and the references herein. In addition, Spearman's rho often performs better in terms of robustness than the Bravais-Pearson correlation. Several other pitfalls and possible problems for a risk manager who simply applies the Bravais-Pearson correlation are discussed in Embrechts et al. (2002).

Therefore it is natural in the context of testing for changes in the dependence structure of random vectors to extend the BPC test to a test for constant Spearman's rho. As expected from the theory of dependence measures, this test is applicable in more situations: It has a much better behavior in the presence of outliers and there are no conditions on the existence of moments, while the BPC test (as well as Aue et al., 2009) requires finite fourth moments. In addition, the test is applicable in arbitrary dimensions, while the BPC test is designed for bivariate random vectors. Similarly to the BPC test, the test bases on successively calculated empirical correlation coefficients in the vein of Ploberger et al. (1989) or Lee et al. (2003) and the limit distribution of our test statistic is the supremum of the absolute value of a Brownian Bridge. This immediately provides critical values without any bootstrap techniques. We impose a strong mixing assumption for the dependence structure. The proof relies on an invariance principle for multivariate sequential empirical processes from Rüschendorf (1976).

By using the copula-based expression for Spearman's rho from Schmid and Schmidt (2007) or Nelson (2006), we get quite another contribution with our test, i.e. an extension of the copula constancy tests proposed by Krämer and van Kampen (2010), Busetti and Harvey (2011) or van Kampen and Wied (2010). These tests are important in financial econometrics, but are restricted to the case of testing for copula constancy in one particular quantile, e.g. the 0.95-quantile. This might be an important null hypothesis as well, but our test now allows for testing constancy of the whole copula by integrating over it. We therefore reject the null hypothesis of constant Spearman's rho (which is closely connected to the null hypothesis of an overall constant copula) if the integral over it fluctuates too much over time.

The paper is organized as follows: Section 2 presents our test statistic and the asymptotic null distribution, Section 3 considers local power, Section 4 presents Monte Carlo evidence about the test's behavior in small samples and Section 5 compares our new test with the BPC test in terms of robustness by a simulation study and an empirical application. Finally, Section 6 concludes. All proofs are in the appendix.

2. Test statistic and its asymptotic null distribution

In this section, we present the test statistic and the limit distribution of our test under the null. First, we want to introduce notation: $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ are d-dimensional random vectors with $\mathbf{X}_j = (X_{1,j}, \ldots, X_{d,j})$. Regarding the dependence structure, the following assumption is imposed:

(A1) $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are α-mixing with mixing coefficients α_j satisfying

$$\sum_{j=1}^{\infty} j^2 \alpha_j^{\gamma/(4+\gamma)} < \infty$$

for some $\gamma \in (0, 2)$.

This dependence assumption holds in most of the econometric models relevant in practice, see e.g. Inoue (2001).

The vectors $\mathbf{X}_j, j = 1, \dots, n$, have joint distribution functions F^j with

$$F^j(\mathbf{x}) = \mathsf{P}(X_{1,j} \leq x_1, \dots, X_{d,j} \leq x_d), \mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d,$$

and marginal distribution functions $F_{i,j}(x) = \mathsf{P}(X_{i,j} \leq x)$ for $x \in \mathbb{R}$ and $i = 1, \dots, d$.

With Sklar's (1959) theorem, there exists a unique copula function $C_j : [0,1]^d \to [0,1]$ of $\mathbf{X}_j$ with

$$F^j(\mathbf{x}) = C_j(F_{1,j}(x_1), \dots, F_{d,j}(x_d))$$

and

$$C_j(\mathbf{u}) = F_j(F_{1,j}^{-1}(u_1), \dots, F_{d,j}^{-1}(u_d)), \mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d.$$

In terms of the copula, Spearman's rho is defined as

$$\rho_j = h(d) \cdot \left(2^d \int_{[0,1]^d} C_j(\mathbf{u}) d\mathbf{u} - 1 \right)$$

with

$$h(d) = \frac{d+1}{2^d - (d+1)},$$

see Schmid and Schmidt (2007) or Nelson (2006).

We are testing

$$H_0 : \rho_j = \rho_0, j = 1, \dots, n \text{ vs. } H_1 : \exists j \in \{1, \dots, n-1\} : \rho_j \neq \rho_{j+1}.$$

Let

$$\hat{F}_{i;n}(x) = \frac{1}{n}\sum_{j=1}^{n} \mathbf{1}_{\{X_{i,j}\le x\}}, i = 1, \dots, d, x \in \mathbb{R},$$

$U_{i,j} = F_{i,j}(X_{i,j})$ and

$$\hat{U}_{i,j;n} := \hat{F}_{i;n}(X_{i,j}) = \frac{1}{n}\left(\text{rank of } X_{i,j} \text{ in } X_{i,1}, \dots, X_{i,n}\right), i = 1, \dots, d, j = 1, \dots, n.$$

Let

$$R_j(\mathbf{u}) = \mathbf{1}_{\{U_{1,j}\le u_1,\dots,U_{d,j}\le u_d\}}$$

and

$$\hat{R}_j(\mathbf{u}) = \mathbf{1}_{\{\hat{U}_{1,j;n}\le u_1,\dots,\hat{U}_{d,j;n}\le u_d\}}.$$

The copula C is estimated by the empirical copula, defined as

$$\hat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(\mathbf{u}) = \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}\mathbf{1}_{\{\hat{U}_{i,j;n}\le u_i\}}, \mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d.$$

The estimator based on the first k observations is

$$\hat{C}_k(\mathbf{u}) = \frac{1}{k}\sum_{j=1}^{k}\hat{R}_j(\mathbf{u}) = \frac{1}{k}\sum_{j=1}^{k}\prod_{i=1}^{d}\mathbf{1}_{\{\hat{U}_{i,j;n}\le u_i\}}, \mathbf{u} = (u_1, \dots, u_d) \in [0,1]^d.$$

Note that we must use $\hat{U}_{i,j;n}$ and not $\hat{U}_{i,j;k}$.

The estimator for the copula immediately yields an estimator for Spearman's rho:

$$\hat{\rho}_k = h(d)\cdot\left(2^d\int_{[0,1]^d}\hat{C}_k(\mathbf{u})d\mathbf{u} - 1\right) = h(d)\cdot\left(\frac{2^d}{k}\sum_{j=1}^{k}\prod_{i=1}^{d}(1-\hat{U}_{i,j;n}) - 1\right)$$

We use the following test statistic W:

$$\begin{aligned}
W &= \hat{D} \max_{1\le k\le n} \left| \frac{k}{\sqrt{n}} \left(\hat{\rho}_k - \hat{\rho}_n\right) \right| \\
&= \hat{D} \sup_{s\in[0,1]} \left| \frac{[ns]}{\sqrt{n}} \left(\hat{\rho}_{[ns]} - \hat{\rho}_n\right) \right| \\
&=: \hat{D} \sup_{s\in[0,1]} |P_n(s)|
\end{aligned}$$

with an estimator $\hat{D}$,

$$\hat{D} = \frac{1}{\sqrt{\hat{D}'}},$$

where

$$\begin{aligned}
\hat{D}' = h(d)^2 2^{2d} \Bigg\{ & \frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\hat{U}_{i,j;n})^2 - \left(\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\hat{U}_{i,j;n})\right)^2 \\
&+2\left[\sum_{m=1}^{\gamma_n} k\left(\frac{m}{\gamma_n}\right)\left(\sum_{j=1}^{n-m}\frac{1}{n}\prod_{i=1}^{d}(1-\hat{U}_{i,j;n})(1-\hat{U}_{i,j+m;n}) - \left(\frac{1}{n}\sum_{j=1}^{n}\prod_{i=1}^{d}(1-\hat{U}_{i,j;n})\right)^2\right)\right]\Bigg\}.
\end{aligned}$$

The kernel $k(\cdot)$ fulfills $k(x) = 0$ for $|x| > 1$ and is contained in the class $\mathcal{K}_2$ of de Jong and Davidson (2000) which guarantees positive semi-definiteness (we can choose e.g. the Bartlett-kernel); the bandwidth γ_n is chosen such that $\gamma_n = o(n^{\frac{1}{2}})$.

For the central theorem, we need several additional assumptions:

(B1) $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ is strictly stationary.

(B2) The marginal distribution functions $F_{i,j} = F_i, i = 1, \ldots, d$ are continuous.

(B3) The marginals of C are strictly increasing.

(B4) The partial derivatives $\frac{\partial C}{\partial u_i}(\mathbf{u})$ exist and are continuous for $i = 1, \ldots, d$.

Theorem 1. *Under H_0 and Assumptions (A1), (B1)-(B4),*

$$W \to_d \sup_{s\in[0,1]} |B(s)|,$$

where $B(s)$ is a one-dimensional Brownian Bridge.

Theorem 1 allows for constructing an asymptotic test. The main tool for the proof which can be found in Appendix A is an invariance principle for the multivariate sequential empirical process

$$\begin{aligned} L_n(s,\mathbf{u}) &= \frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}(\hat{R}_j(\mathbf{u}) - C(\mathbf{u})) \\ &= \frac{[ns]}{\sqrt{n}}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(\mathbf{u}) - C(\mathbf{u})\right), \end{aligned}$$

see Rüschendorf (1976).

There exists an interesting relationship between our test for constancy of Spearman's rho and the copula constancy tests proposed by Busetti and Harvey (2011) and van Kampen and Wied (2010): One can show (see Appendix B) that our test is as a functional of the multivariate τ-quantics on which these copula constancy tests base. But, in fact, while the other tests examine if the copula in a particular quantile is constant, we can test for constancy of the whole copula by integrating over it. Note that the integral of the copula is just one particular functional of it (which has of course nice properties as it e.g. leads to a limit distribution which is free of nuisance parameters). Currently, we are trying to extend our approach to other functionals of the copula which lead to other dependence measures as e.g. Kendall's tau (see Nelson, 2006).

3. Local power

This section considers the local power of our test. Since the copula function of the random vectors under consideration changes with n, we now operate with a triangular array $(\mathbf{X}_1^n, \ldots, \mathbf{X}_n^n)$, but we suppress the index n for ease of exposition. Let $C(\mathbf{u})$ be a copula and let $C^*(s, \mathbf{u})$ be another copula with an additional index parameter s. We consider local alternatives of the type

$$C_j(\mathbf{u}) = \left(1 - \frac{\delta}{\sqrt{n}}\right) C(\mathbf{u}) + \frac{\delta}{\sqrt{n}} C^*\left(\frac{j}{n}, \mathbf{u}\right). \tag{1}$$

By choosing, say, $C^*(s, \mathbf{u}) = [1 - g(s)]\, C(\mathbf{u}) + g(s) C^{**}(\mathbf{u})$ for some copula $C^{**}(\cdot)$ and some function $g(\cdot)$ bounded by 1 we obtain the sequence of correlations

$$\rho_j = \left[1 - \frac{\delta}{\sqrt{n}} g\left(\frac{j}{n}\right)\right] \rho_0 + \frac{\delta}{\sqrt{n}} g\left(\frac{j}{n}\right) \rho_A.$$

To deduce limit results for the sequence of local alternatives (1), we need some more assumptions

(C1) The analogue mixing condition (A1) holds for the triangular array.

(C2) The joint copula for the random vectors $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ with lag l,

$$C_{j,l}(\mathbf{u}, \mathbf{v}) := \mathsf{P}(X_{1,j} \leq F_{1,j}^{-1}(u_1), \ldots, X_{d,j} \leq F_{d,j}^{-1}(u_d), X_{1,j+l} \leq F_{1,j+l}^{-1}(v_1), \ldots, X_{d,j+l} \leq F_{d,j+l}^{-1}(v_d)),$$

is specified to

$$\begin{aligned} C_{j,l}(\mathbf{u}, \mathbf{v}) = &\left(1 - \frac{\delta}{\sqrt{n}}\right)^2 C_l(\mathbf{u}, \mathbf{v}) + \frac{\delta^2}{n} C_l^*\left(\frac{j}{n}, \frac{j+l}{n}, \mathbf{u}, \mathbf{v}\right) \\ &+ \frac{\delta}{\sqrt{n}}\left(1 - \frac{\delta}{\sqrt{n}}\right)\left[C(\mathbf{u}) C^*\left(\frac{j+l}{n}, \mathbf{v}\right) + C(\mathbf{v}) C^*\left(\frac{j}{n}, \mathbf{u}\right)\right] \end{aligned}$$

with a constant $\delta \in (0, 1]$. In this equation $C_l(\cdot, \cdot)$ is the joint copula of some sequence of stationary random vectors ξ_i with lag l, $C(\cdot)$ is the copula of ξ_i. Analogously, $C_l^*(\cdot, \cdot, \cdot, \cdot)$ is the copula of some sequence of stationary random functions $\eta_i(\cdot)$ with lag l, $C^*(\cdot, \cdot)$ is the copula of $\eta_i(\cdot)$.

(C3) The marginals of C are strictly increasing.

(C4) The partial derivatives $\frac{\partial C}{\partial u_i}(\mathbf{u})$ and $\frac{\partial C^*}{\partial u_i}(s, \mathbf{u})$ exist and are continuous for $i = 1, \ldots, d$ and all s.

(C5) The marginal distribution functions of $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ do not depend on j and are continuous.

Assumption (B1) is in line with Inoue (2001) and yields equation (1) by letting $\mathbf{v}$ tend to ∞. The equation describes a mixture of two distribution functions and allows for multiple-change local alternatives. With this, we get

Theorem 2. *Under Assumptions (C1) - (C5),*

$$W \to_d \sup_{s \in [0,1]} \left| B(s) + \delta D h(d) 2^d \left[\int_{[0,1]^d} \int_0^s C^*(t, \mathbf{u})\, dt d\mathbf{u} - s \int_{[0,1]^d} \int_0^1 C^*(t, \mathbf{u})\, dt d\mathbf{u} \right] . \right|,$$

where D is the probability limit of $\hat{D}$ under the null hypothesis.

With this theorem and Anderson's Lemma we can deduce that the asymptotic level is always larger than or equal to α, see Andrews (1997) or Rothe and Wied (2011).

4. Finite sample behavior

We investigate the test's finite sample behavior and compare it to the BPC test by simulating the empirical size under the null hypothesis and the empirical power under various alternatives. In all our simulations we use the Bartlett kernel and bandwidth $[\log(n)]$ both for our test and for the BPC test. For serial dependence, we assume a bivariate $MA(1)$-process

$$\mathbf{X}_t = \epsilon_t + \theta \epsilon_{t-1} \text{ with } \theta = \begin{pmatrix} \theta_1 = 0.3 & 0 \\ 0 & \theta_2 = 0.2 \end{pmatrix}.$$

In this situation the $\epsilon_t, t \in \mathbb{Z}$ are independent and identically distributed, following the t_1-distribution with shape matrix

$$S = \begin{pmatrix} 1 & s \\ s & 1 \end{pmatrix}, \ |s| < 1. \tag{2}$$

In this case, we do not have finite fourth moments (even no finite first moment) which are required for the BPC test. The null hypothesis is that $\mathbf{X}_t$ has constant correlation of $\rho_0 = 0.4$. Additionally we consider six alternatives A1 to A6, in which the correlation jumps after the middle of the sample from $\rho_0 = 0.4$ to $\rho_1 = 0.6, 0.8, 0.2, 0.0, -0.2, -0.4, -0.6$, respectively. For the simulations, we generate realizations $\epsilon_0, \epsilon_1, \ldots, \epsilon_{n/2}$ and $\epsilon_{n/2+1}, \ldots, \epsilon_n$ with $s_i = \rho_i \sqrt{\frac{(\theta_1^2+1)(\theta_2^2+1)}{\theta_1\theta_2+1}}, i = 0, 1$.[2] Table 1 reports rejection frequencies at the significance level $\alpha = 0.05$ based on 5000 repetitions and sample sizes $n = 500, 1000, 2000$.

-Table 1 here -

We see that the size of our is kept and that the empirical power increases with the magnitude and n. It is slightly higher for increasing than for decreasing correlations, what is important in risk management. The BPC test is not applicable at all, because it cannot distinguish between null hypothesis and alternative. This is an expected behavior considering that the asymptotic variance of the empirical correlation coefficient is an unbounded function of the fourth moments of the population distribution and that, on the other hand, Spearman's correlation coefficient is invariant under monotonely increasing, componentwise transformations and hence little effected by heavy tails.

We get a partly similar result when using the setup from above with the t_5-distribution, constant correlation 0.4 and by adding one heavy outlier of size, say, $(40, -100)$ to the sample at time $c \cdot n, c = 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.45, 0.5$. If the outlier comes late, the BPC test almost always rejects the null hypothesis; if it comes early, the test

[2] The choice of the s_i is due to the fact that with this, the Pearson correlation would be equal to ρ_0 resp. ρ_1 if it existed. Spearman's rho lies then closely to these values as numerical approximations suggest.

almost never rejects it. This is an odd behavior and makes the test unsuitable for this outlier scenario. Our new test always keeps the size, see Table 2.

-Table 2 here -

In a setup for distributions with lower tails and no outliers, the efficiency of our copula-based test is low compared to the BPC test, see Table 3 for exemplary results for the t_3-distribution. Note that the t_3 is already "closer" to the Gaussian distribution where the usual empirical correlation coefficient is the maximum likelihood, i.e. the most efficient estimator of the correlation.

-Table 3 here -

5. Robustness

5.1. Simulation evidence The two major advantages of our test compared to the BPC test proposed by Wied et al. (2010) are its applicability without any moment conditions at all (as compared to the existence of fourth moments for the BPC test) and its appealing robustness properties. The latter shall be visualized by an instructive example. Both proposed fluctuation tests mainly derive their robustness properties from the respective properties of the underlying correlation measure. The robustness properties of Spearman's rho, along with several other correlation estimators, are studied in detail in Croux and Dehon (2010).

We sample a path $(\mathbf{x}_t)_{t=1,\ldots,500}$ of length $n = 500$ of the bivariate MA(1) process $\mathbf{X}_t = \epsilon_t + \theta\epsilon_{t-1}$, where the ϵ_t, $t \in \mathbb{Z}$, are i.i.d. with a centered bivariate Gaussian distribution and covariance matrix S. The parameters θ and S are as in Section 4, where the correlation s of ϵ_t is, as under the null before, chosen such that the resulting correlation of $\mathbf{X}_t$ is 0.4. We add one mild outlier to the sample by setting, say, $\mathbf{x}_{288}$ to $(20, -50)$. We denote the resulting contaminated example by $(\mathbf{x}_t^w)_{t=1,\ldots,n}$, where w indicates *w*eak contamination.

Figure 1 visualizes the process

$$b_k = \tilde{D} \frac{k}{\sqrt{n}} (\hat{r}_k - \hat{r}_n), \qquad k = 1, ..., n,$$

where $\hat{r}_k$ denotes the Bravais-Pearson correlation coefficient based on $\mathbf{X}_1, ..., \mathbf{X}_k$, and $\tilde{D}$ is a deviation estimator that is described in Wied et al. (2010) and scales the process such that $(b_{[ns]})_{s\in[0,1]}$ converges to a Brownian bridge.

-Figure 1 here -

-Figure 2 here -

The BPC test statistic is then $\sup_{1\leq k\leq n} |b_k|$. The grey line in Figure 1 corresponds to the uncontaminated sample $(\mathbf{x}_t)_{t=1,...,n}$, the black line to $(\mathbf{x}_t^w)_{t=1,...,n}$. The single outlier has a dramatic effect on the Bravais–Pearson test statistic and, in this example, causes the null hypothesis to be rejected at the significance level 0.05. The results are similar if the outlier is placed at a different position.

Alternatively we create a strongly contaminated sample $(\mathbf{x}_t^s)_{t=1,...,n}$ by randomly placing 10 outliers in the second half of the sample. Each outlier is of the form $(y_t, -y_t)$, where y_t is drawn from the uniform distribution on $[-1000, -100] \cup [100, 1000]$. Both, fraction and size of the outliers in $(\mathbf{x}_t^s)_{t=1,...,n}$ are about 10 times as large as in $(\mathbf{x}_t^w)_{t=1,...,n}$. Figure 2 depicts the process

$$s_k = \hat{D} \frac{k}{\sqrt{n}} (\hat{\rho}_k - \hat{\rho}_n), \qquad k = 1, ..., n,$$

once being computed from the uncontaminated sample (grey line) and once from the heavily corrupted sample $(\mathbf{x}_t^s)_{t=1,...,n}$ (black line). We witness a slight distortion of $(s_k)_{k=1,...,n}$ as a result of the contamination, but the location of the maximizing point as well as the decision of the test are unaffected. Similar results are obtained for other realizations.

5.2. Empirical relevance This subsection shows that the outlier scenario described in the previous subsection might indeed be relevant for a practitioner who analyzes structural changes in the dependence of assets. This can be exemplarily seen in the time period around the Black Monday, 19th October 1987, i.e. in the time from January 1985 to December 1989. Both the Dow Jones Industrial Average and the Nasdaq Composite daily returns are extremely negative at this day and the absolute values of these returns are much higher than the other ones at this time. One day after, the Dow Jones return is positive again, while the Nasdaq return remains negative - both on a high level compared to the means and standard deviations of all days. Table 4 shows the exact values.

-Table 4 here -

These outliers are reflected in the BPC test statistic, see Figure 3, part (a), for the weighted differences of successively estimated Pearson correlation coefficients and the peak around the Black Monday. On 19th October, the successively estimated correlations become very high, but due to the behavior one day after, they fall down immediately. Both phenomena together lead to the peak.[3]

Applying the BPC test gives a test statistic of 1.447 (p-value of 0.030) such that the null hypothesis of constant correlation is rejected on the 5%-level, but the test statistic would be much lower without this peak and the null would not be rejected.

Our Spearman test statistic is not affected by this peak - see Figure 3, part (b) - and the test statistic is equal to 0.886 (p-value of 0.412). Therefore, one should probably conclude that the dependence structure did not change seriously after the Black Monday. Similar results were obtained for other time periods around 19th October 1987.

-Figure 3 here -

[3] This is not exactly the same situation as in Table 1. However, by similar simulations with two outliers as in this applications we can reproduce the peak of Figure 3 (a) as well. Thus, the figures give two different examples of a bizarre behavior of the test statistic which are both due to the construction of the Pearson correlation coefficient.

6. Discussion

We have proposed a new test for constancy of Spearman's rho which is much more robust against outliers than the BPC test previously suggested by Wied et al. (2010).

Our test also allows for testing if the whole copula of multivariate random vectors is constant, therefore extending former suggested copula constancy tests. It is a task for further research to extend this test to other functionals of the copula in order to check if the performance might be better for certain alternatives. Another task for further research is the extension of the dependence structure to functionals of iid- or even of mixing processes to enlarge the class of models in which our test can operate.

A. Appendix section

Proof of Theorem 1

Consider first $P_n(s)$:

$$\begin{aligned} P_n(s) &= \frac{[ns]}{\sqrt{n}}\left(h(d)\cdot\left(2^d\int_{[0,1]^d}\hat{C}_{[ns]}(\mathbf{u})d\mathbf{u}-1\right)-h(d)\cdot\left(2^d\int_{[0,1]^d}\hat{C}_n(\mathbf{u})d\mathbf{u}-1\right)\right) \\ &= \frac{[ns]}{\sqrt{n}}\cdot h(d)\cdot 2^d\int_{[0,1]^d}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(\mathbf{u})-\frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(\mathbf{u})\right)d\mathbf{u} \\ &= h(d)\cdot 2^d\int_{[0,1]^d}A_n(s,\mathbf{u})d\mathbf{u} \end{aligned}$$

with

$$A_n(s,\mathbf{u}) = \frac{[ns]}{\sqrt{n}}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(\mathbf{u})-\frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(\mathbf{u})\right). \tag{3}$$

Now,

$$\begin{aligned} A_n(s,\mathbf{u}) &= \frac{[ns]}{\sqrt{n}}\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(\mathbf{u}) - \frac{[ns]}{\sqrt{n}}C(\mathbf{u}) - \frac{[ns]}{\sqrt{n}}\frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(\mathbf{u}) + \frac{[ns]}{\sqrt{n}}C(\mathbf{u}) \\ &= \frac{[ns]}{\sqrt{n}}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(\mathbf{u}) - C(\mathbf{u})\right) - \frac{[ns]}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(\mathbf{u}) - C(\mathbf{u})\right) \\ &= L_n(s,\mathbf{u}) - \frac{[ns]}{n}L_n(1,\mathbf{u}). \end{aligned}$$

With Theorem 3.3 in Rüschendorf (1976) it holds under the assumptions

$$A_n(\cdot,\cdot) \to_d A_0(\cdot,\cdot),$$

where

$$A_0(s,\mathbf{u}) = L_0(s,\mathbf{u}) - sL_0(1,\mathbf{u})$$

with

$$L_0(s,\mathbf{u}) = V_0(s,\mathbf{u}) - s\sum_{i=1}^{d}\frac{\partial C}{\partial u_i}(\mathbf{u})V_0(1,1,\ldots,u_i,\ldots,1).$$

The convergence of $A_n(\cdot,\cdot)$ follows from the convergence of $L_n(\cdot,\cdot)$; for this, we need an invariance principle for

$$\frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}(R_j(\mathbf{u}) - C(\mathbf{u})). \tag{4}$$

The latter is presented in Inoue (2001) in a slightly more general form than we need it here.

$V_0(s,\mathbf{u})$ is a P-almost surely continuous, centered Gaussian process with covariance function

$$K_0((s_1,\mathbf{u_1}),(s_2,\mathbf{u_2})) := \mathsf{Cov}(V_0(s_1,\mathbf{u_1}),V_0(s_2,\mathbf{u_2})) = (s_1\wedge s_2)K'(\mathbf{u_1},\mathbf{u_2})$$

for

$$\begin{aligned}K'(\mathbf{u_1},\mathbf{u_2}) &= C(\mathbf{u_1}\wedge\mathbf{u_2}) - C(\mathbf{u_1})C(\mathbf{u_2})\\ &\quad+\sum_{m=2}^{\infty}\Big(\mathsf{E}(\mathbf{1}_{\{X_{1,1}\le F_1^{-1}(u_1^1);\dots;X_{d,1}\le F_d^{-1}(u_1^d)\}}\mathbf{1}_{\{X_{1,m}\le F_1^{-1}(u_2^1);\dots;X_{d,m}\le F_d^{-1}(u_2^d)\}})\\ &\quad-\mathsf{E}(\mathbf{1}_{\{X_{1,1}\le F_1^{-1}(u_1^1);\dots;X_{d,1}\le F_d^{-1}(u_1^d)\}})\cdot\mathsf{E}(\mathbf{1}_{\{X_{1,m}\le F_1^{-1}(u_2^1);\dots;X_{d,m}\le F_d^{-1}(u_2^d)\}})\\ &\quad+\mathsf{E}(\mathbf{1}_{\{X_{1,1}\le F_1^{-1}(u_2^1);\dots;X_{d,1}\le F_d^{-1}(u_2^d)\}}\mathbf{1}_{\{X_{1,m}\le F_1^{-1}(u_1^1);\dots;X_{d,m}\le F_d^{-1}(u_1^d)\}})\\ &\quad-\mathsf{E}(\mathbf{1}_{\{X_{1,1}\le F_1^{-1}(u_2^1);\dots;X_{d,1}\le F_d^{-1}(u_2^d)\}})\cdot\mathsf{E}(\mathbf{1}_{\{X_{1,m}\le F_1^{-1}(u_1^1);\dots;X_{d,m}\le F_d^{-1}(u_1^d)\}})\Big).\end{aligned}$$

This covariance function is the limit of the covariance function of

$$V_n(s,\mathbf{u}) = \frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}(R_j(\mathbf{u}) - C(\mathbf{u}))$$

i.e.

$$K_0((s_1,\mathbf{u_1}),(s_2,\mathbf{u_2})) = \lim_{n\to\infty}\mathsf{Cov}(V_n(s_1,\mathbf{u_1}),V_n(s_2,\mathbf{u_2})).$$

It follows

$$A_0(s,\mathbf{u}) = V_0(s,\mathbf{u}) - sV_0(1,\mathbf{u}).$$

With the Continuous Mapping Theorem,

$$P_n(\cdot)\to_d P_0(\cdot),$$

where

$$P_0(s) = h(d)2^d \int_{[0,1]^d} A_0(s, \mathbf{u}) d\mathbf{u}$$

is a P-almost surely continuous, centered Gaussian process. With Fubini's theorem, the covariance function is

$$\begin{aligned}
&\mathsf{Cov}(P_0(s_1), P_0(s_2)) \\
&= h(d)^2 2^{2d} \int_{[0,1]^d} \int_{[0,1]^d} \mathsf{Cov}(V_0(s_1, \mathbf{u}) - s_1 V_0(1, \mathbf{u}), V_0(s_2, \mathbf{v}) - s_2 V_0(1, \mathbf{v})) d\mathbf{u} d\mathbf{v} \\
&= h(d)^2 2^{2d} (s_1 \wedge s_2 - s_1 s_2 - s_1 s_2 + s_1 s_2) \int_{[0,1]^d} \int_{[0,1]^d} \mathsf{Cov}(V_0(1, \mathbf{u}), V_0(1, \mathbf{v})) d\mathbf{u} d\mathbf{v} \\
&= (s_1 \wedge s_2 - s_1 s_2) D'
\end{aligned}$$

with

$$\begin{aligned}
D' &= h(d)^2 2^{2d} \int_{[0,1]^d} \int_{[0,1]^d} K'(\mathbf{u}, \mathbf{v}) d\mathbf{u} d\mathbf{v} \\
&= h(d)^2 2^{2d} \left\{ \mathsf{E}\left(\prod_{i=1}^d (1 - U_{i,j})^2 \right) - \left[\mathsf{E}\left(\prod_{i=1}^d (1 - U_{i,j}) \right) \right]^2 \right. \\
&\left. +2 \left[\sum_{m=1}^{\infty} \mathsf{E}\left(\prod_{i=1}^d (1 - U_{i,j})(1 - U_{i,j+m}) \right) - \left(\mathsf{E}\left(\prod_{i=1}^d (1 - U_{i,j}) \right) \right)^2 \right] \right\} \\
&= h(d) 2^{2d} \left[\mathsf{Var}\left(\prod_{i=1}^d (1 - U_{i,j}) \right) + 2 \sum_{m=1}^{\infty} \mathsf{Cov}\left(\prod_{i=1}^d (1 - U_{i,j}), \prod_{i=1}^d (1 - U_{i,j+m}) \right) \right]
\end{aligned}$$

This holds because, again with Fubini,

$$
\begin{aligned}
&\int_{[0,1]^d}\int_{[0,1]^d} C(\mathbf{u}\wedge\mathbf{v})d\mathbf{u}d\mathbf{v} \\
&= \int_{[0,1]^d}\int_{[0,1]^d} \mathsf{E}(\mathbf{1}_{\{X_{1,j}\le F_1^{-1}(u_1^1);\dots;X_{d,j}\le F_d^{-1}(u_1^d)\}}\mathbf{1}_{\{X_{1,j}\le F_1^{-1}(u_2^1);\dots;X_{d,j}\le F_d^{-1}(u_2^d)\}})d\mathbf{u}d\mathbf{v} \\
&= \mathsf{E}\left(\int_{[0,1]^d}\int_{[0,1]^d} \mathbf{1}_{\{X_{1,j}\le F_1^{-1}(u_1^1);\dots;X_{d,j}\le F_d^{-1}(u_1^d)\}}\mathbf{1}_{\{X_{1,j}\le F_1^{-1}(u_2^1);\dots;X_{d,j}\le F_d^{-1}(u_2^d)\}}d\mathbf{u}d\mathbf{v}\right) \\
&= \mathsf{E}\left(\int_{[0,1]^d}\int_{[0,1]^d} \mathbf{1}_{\{U_{1,j}\le u_1^1;\dots;U_{d,j}\le u_1^d\}}\mathbf{1}_{\{U_{1,j}\le u_2^1;\dots;U_{d,j}\le u_2^d\}}d\mathbf{u}d\mathbf{v}\right) \\
&= \mathsf{E}\left(\prod_{i=1}^d(1-U_{i,j})\prod_{i=1}^d(1-U_{i,j})\right).
\end{aligned}
$$

The other summands of $K'(\mathbf{u},\mathbf{v})$ are integrated analoguesly.

We get a consistent estimator for D' from de Jong and Davidson (2000),

$$
\begin{aligned}
\tilde{D}' = h(d)^2 2^{2d} &\left\{ \frac{1}{n}\sum_{j=1}^n\prod_{i=1}^d(1-U_{i,j})^2 - \left(\frac{1}{n}\sum_{j=1}^n\prod_{i=1}^d(1-U_{i,j})\right)^2\right. \\
&\left. +2\left[\sum_{m=1}^{\gamma_n} k\left(\frac{m}{\gamma_n}\right)\left(\sum_{j=1}^{n-m}\frac{1}{n}\prod_{i=1}^d(1-U_{i,j})(1-U_{i,j+m}) - \left(\frac{1}{n}\sum_{j=1}^n\prod_{i=1}^d(1-U_{i,j})\right)^2\right)\right]\right\},
\end{aligned}
$$

with a kernel k that fulfills the condition $k(x)=0$ for $|x|>1$ and is contained in the class $\mathcal{K}_2$ of de Jong and Davidson (2000) which guarantees positive semi-definiteness of $\tilde{D}'$. Next, we show that $\hat{D}'-\tilde{D}'\to_p 0$. By the invariance principle (4) we get a Gliwenko-Cantelli-like theorem (in probability) with rate $n^{-\frac{1}{2}}$, that means,

$$
B_n := \max_{i=1,\dots,d}\sup_{j\in\mathbb{N}}|\hat{U}_{i,j;n}-U_{i,j}| = O_{\mathsf{P}}\left(n^{-\frac{1}{2}}\right).
$$

Since $0\le U_{i,j},\hat{U}_{i,j;n},\le 1$, we obtain

$$
\left|\prod_{i=1}^d(1-\hat{U}_{i,j;n})(1-\hat{U}_{i,j+m;n}) - \prod_{i=1}^d(1-U_{i,j})(1-U_{i,j+m})\right| \le 2dB_n.
$$

Thus we get

$$|\hat{D}' - \tilde{D}| \leq C \sum_{m=1}^{\gamma_n} k\left(\frac{m}{\gamma_n}\right) B_n = O\left(\gamma_n n^{-\frac{1}{2}}\right) = o_{\mathsf{P}}(1),$$

as $\gamma_n = o(n^{\frac{1}{2}})$. Therefore $\hat{D}'$ is a consistent estimator of D'.

The theorem follows then with the Continuous Mapping Theorem, because the process

$$P_0^*(s) := \frac{1}{\sqrt{D'}} P_0(s)$$

is a P-almost surely continuous, centered Gaussian process with the same covariance function as the Brownian Bridge, i.e.

$$\mathsf{Cov}(P_0^*(s_1), P_0^*(s_2)) = s_1 \wedge s_2 - s_1 s_2.$$

Since a Gaussian process is uniquely determined by the first two moments, the limit process is in fact a Brownian Bridge. ■

Proof of Theorem 2

We consider $A_n(s, \mathbf{u})$ from (3). By adding suitable averages over $C_j(\mathbf{u})$ it holds

$$\begin{aligned} A_n(s,\mathbf{u}) = {} & \frac{[ns]}{\sqrt{n}} \left(\frac{1}{[ns]} \sum_{j=1}^{[ns]} (\hat{R}_j(\mathbf{u}) - C_j(\mathbf{u})) \right) - \frac{[ns]}{\sqrt{n}} \left(\frac{1}{n} \sum_{j=1}^{n} (\hat{R}_j(\mathbf{u}) - C_j(\mathbf{u})) \right) \\ & + \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} C_j(\mathbf{u}) - \frac{[ns]}{n} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} C_j(\mathbf{u}). \end{aligned}$$

We denote

$$B_n(s,\mathbf{u}) := \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} C_j(\mathbf{u}) - \frac{[ns]}{n} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} C_j(\mathbf{u}).$$

Using similar arguments as in the proof of Theorem 1, i.e. using the analogous invariance principle for

$$V_n(s, \mathbf{u}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[ns]} (R_j(\mathbf{u}) - C_j(\mathbf{u})),$$

which is presented in Inoue (2001), one shows that the first two summands converge to $A_0(s, \mathbf{u})$. Furthermore $B_n(s, \mathbf{u})$ is equal to

$$B_n(s, \mathbf{u}) = \delta \left[\frac{1}{n} \sum_{j=1}^{[ns]} C^* \left(\frac{j}{n}, \mathbf{u} \right) - \frac{[ns]}{n} \frac{1}{n} \sum_{j=1}^{[ns]} C^* \left(\frac{j}{n}, \mathbf{u} \right) \right].$$

This expression converges to

$$\delta \left[\int_0^s C^* (t, \mathbf{u}) \, dt - s \int_0^1 C^* (t, \mathbf{u}) \, dt \right].$$

In addition, the probability limit of $\hat{D}$ under the sequence of local alternatives is the quantity D from the proof of Theorem 1, i.e. the probability limit of $\hat{D}$ under the null hypothesis. This holds because

$$\lim_{n \to \infty} \mathsf{Cov}(V_n(s_1, \mathbf{u_1}), V_n(s_2, \mathbf{u_2}))$$

is the same under the null hypothesis as well as under the sequence of local alternatives. Thus, the theorem is proved. ■

B. Connection to copula constancy tests

Let $\hat{F}_{i,n}^{-1}(u_i)$ denote the empirical quantile function (see e.g. Welsh (1996, p39)). Write

$$\begin{aligned}
A_n(s,u) &= \frac{[ns]}{\sqrt{n}}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{R}_j(u) - \frac{1}{n}\sum_{j=1}^{n}\hat{R}_j(u)\right)\\
&= -\frac{[ns]}{\sqrt{n}}\left(\frac{1}{[ns]}\sum_{j=1}^{[ns]}\hat{C}_n(u) - \hat{R}_j(u)\right) + \frac{[ns]}{\sqrt{n}}\left(\frac{1}{n}\sum_{j=1}^{n}\hat{C}_n(u) - \hat{R}_j(u)\right)\\
&= -\frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}\Big(\hat{C}_n(u) - \hat{R}_j(u)\Big) + \frac{1}{\sqrt{n}}\sum_{k=1}^{[ns]}\left(\frac{1}{n}\sum_{j=1}^{n}\hat{C}_n(u) - \hat{R}_j(u)\right)\\
&= -\frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}\left[\Big(\hat{C}_n(u) - \hat{R}_j(u)\Big) - \frac{1}{n}\sum_{m=1}^{n}\Big(\hat{C}_n(u) - \hat{R}_m(u)\Big)\right]\\
&= -\frac{1}{\sqrt{n}}\sum_{j=1}^{[ns]}\Big[\hat{C}_n(u) - I(X_{1j} \le \hat{F}_{1,n}^{-1}(u_1), \ldots, X_{dj} \le \hat{F}_{d,n}^{-1}(u_d))\Big]
\end{aligned}$$

where the last step uses that $1/n\sum_{m=1}^{n}\hat{R}_m(u) = \hat{C}_n(u)$ and

$$\begin{aligned}
\hat{R}_j(u) &:= I(\hat{U}_{1j,n} \le u_1, \ldots, \hat{U}_{dj,n} \le u_d)\\
&= I(\hat{F}_{1,n}^{-1}(\hat{U}_{1j,n}) \le \hat{F}_{1,n}^{-1}(u_1), \ldots, \hat{F}_{d,n}^{-1}(\hat{U}_{dj,n}) \le \hat{F}_{d,n}^{-1}(u_d))\\
&= I(X_{1j} \le \hat{F}_{1,n}^{-1}(u_1), \ldots, X_{dj} \le \hat{F}_{d,n}^{-1}(u_d))
\end{aligned}$$

Note that $\hat{C}_n(u) - I(X_{1j} \le \hat{F}_{1,n}^{-1}(u_1), \ldots, X_{dj} \le \hat{F}_{d,n}^{-1}(u_d))$ are the bivariate τ-quantics in Busetti and Harvey (2011) if $d = 2$. Hence, the test can be written as an functional form of the same quantities.

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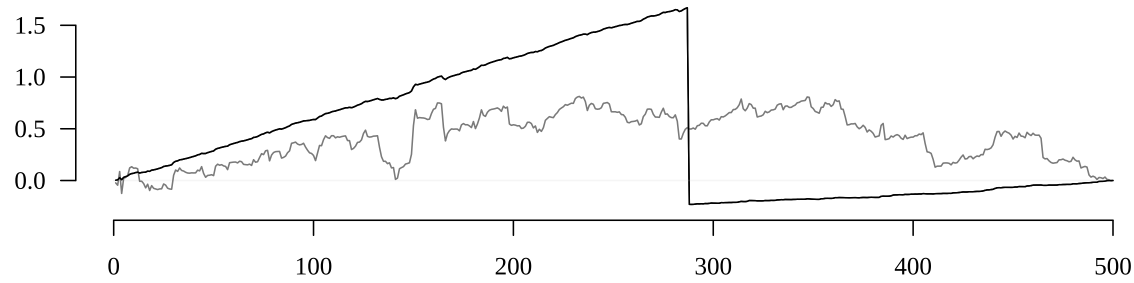


Figure 1: Process $(b_k)_{k=1,...,500}$ computed from weakly contaminated (black) and uncontaminated (grey) sample

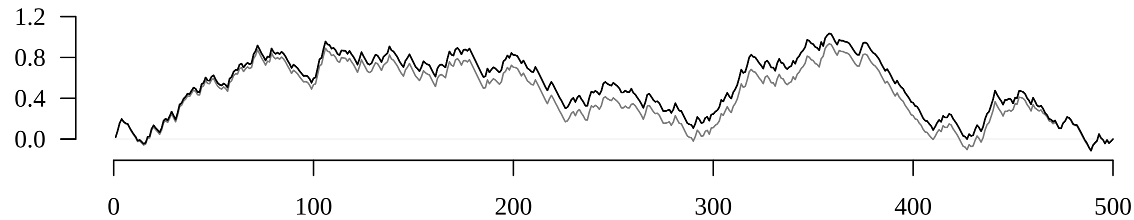


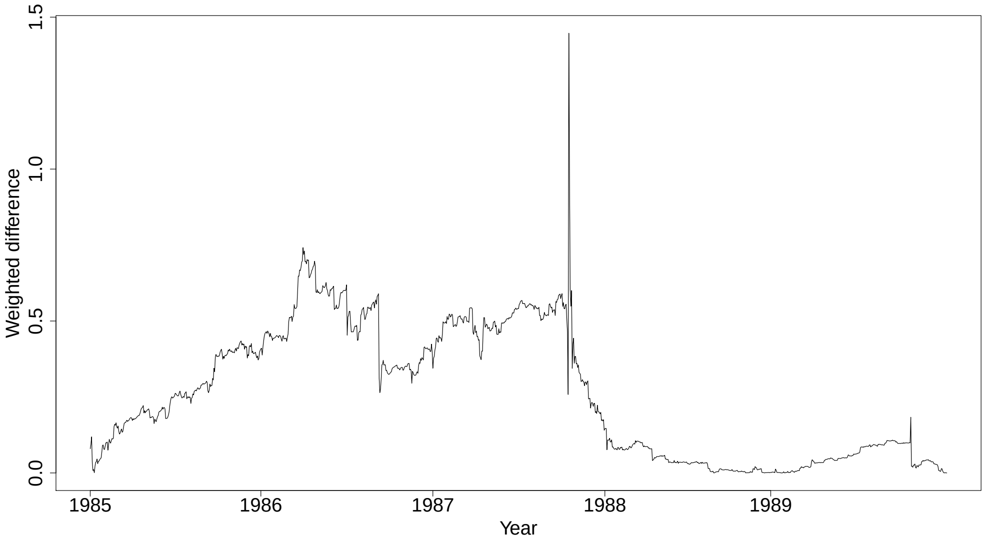
Figure 2: Process $(s_k)_{k=1,...,500}$ computed from strongly contaminated (black) and uncontaminated (grey) sample

n	Values of ρ_1							
	0.4	0.6	0.8	0.2	0	−0.2	−0.4	−0.6
500	0.044	0.067	0.165	0.071	0.138	0.282	0.506	0.740
	(0.489)	(0.514)	(0.560)	(0.480)	(0.481)	(0.502)	(0.523)	(0.540)
1000	0.048	0.102	0.282	0.094	0.256	0.530	0.811	0.969
	(0.479)	(0.512)	(0.552)	(0.474)	(0.496)	(0.497)	(0.514)	(0.544)
2000	0.051	0.160	0.486	0.155	0.466	0.834	0.982	1
	(0.487)	(0.495)	(0.557)	(0.486)	(0.467)	(0.500)	(0.511)	(0.538)

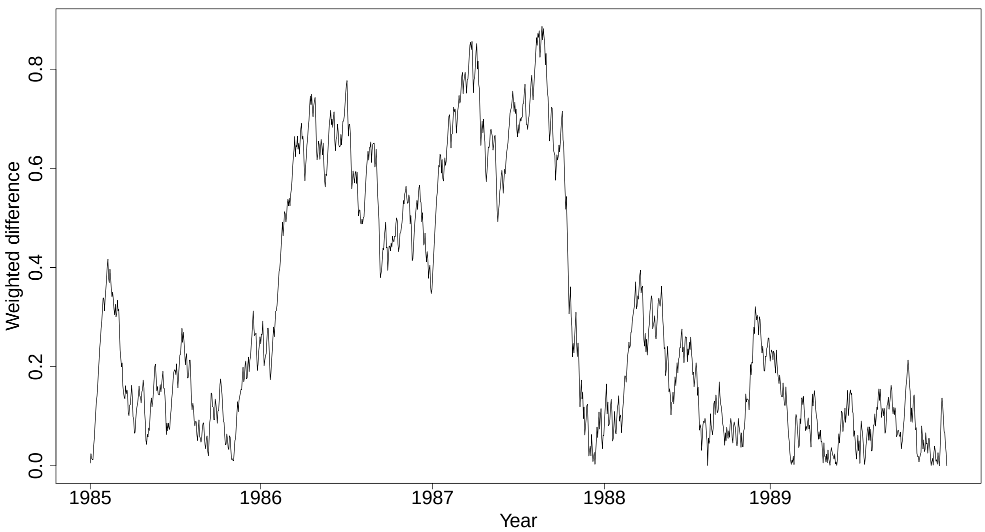
Table 1: Empirical power in different settings for $MA(1)$ serial dependence and the t_1-distribution, $\rho_0 = 0.4$, results of the BPC test in brackets

Fraction outlier point							
0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
0.051	0.051	0.051	0.052	0.051	0.053	0.053	0.053
(0)	(0.001)	(0.190)	(0.820)	(0.980)	(0.996)	(0.999)	(0.999)

Table 2: Empirical rejection frequencies in an outlier scenario, $n = 500$, results of the BPC test in brackets



(a) Process b_k



(b) Process s_k

Figure 3: Processes b_k and s_k for the Dow Jones and Nasdaq Index

n	Values of ρ_1							
	0.4	0.6	0.8	0.2	0	-0.2	-0.4	-0.6
500	0.042	0.080	0.177	0.076	0.176	0.366	0.613	0.854
	(0.056)	(0.287)	(0.725)	(0.143)	(0.422)	(0.700)	(0.852)	(0.912)
1000	0.046	0.107	0.306	0.109	0.323	0.646	0.911	0.992
	(0.045)	(0.358)	(0.833)	(0.204)	(0.601)	(0.845)	(0.919)	(0.946)
2000	0.050	0.175	0.549	0.177	0.567	0.931	0.997	1
	(0.037)	(0.490)	(0.914)	(0.319)	(0.768)	(0.906)	(0.957)	(0.966)

Table 3: Empirical power in different settings for $MA(1)$ serial dependence and the t_3-distribution, $\rho_0 = 0.4$, results of the BPC test in brackets

Day	16.10.	19.10.	20.10.	21.10.	22.10.	$\hat{\mu}$	$\hat{\sigma}$
Dow Jones	-0.047	-0.256	0.057	0.097	-0.039	0	0.013
Nasdaq	-0.039	-0.120	-0.094	0.071	-0.046	0	0.009

Table 4: Dow Jones and Nasdaq Returns around the Black Monday and empirical moments of the whole time span